SAGE PACKAGE *ZONAL.SAGE*

LIN JIU

1. Definition and algorithm

The definition, properties and algorithm here follow from [1, pp. 227–237].

Definition 1. Let λ be a partition of integer n, i.e., $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ such that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq 1 \quad \text{and} \quad \lambda_1 + \lambda_2 + \cdots + \lambda_l = n.$$

The zonal polynomial $\mathcal{C}_\lambda(Y)$ is a symmetric polynomial of degree n, in the eigenvalues $y_1, \ldots, y_m$ of Y, satisfying

$$\sum_{\lambda \in \mathcal{P}_n} \mathcal{C}_\lambda(Y) = (\mathrm{tr}Y)^n = (y_1 + \cdots + y_m)^n. \tag{1.1}$$

Let $\mathcal{P}_n$ be the set of all integer partitions of n. The lexicographical order on $\mathcal{P}_n$ is defined as follows. For any $p = (p_1, \ldots, p_j), q = (q_1, \ldots, q_l) \in \mathcal{P}_n$,

$$p > q \Leftrightarrow p_1 = q_1, \cdots p_{k-1} = q_{k-1}, p_k > q_k \text{ for some } k.$$

Now, another family of symmetric polynomials M, which form a basis to express $\mathcal{C}_\lambda(Y)$, are defined next.

Definition 2. For $\lambda = (\lambda_1, \ldots, \lambda_l) \in \mathcal{P}_n$, define the *monomial symmetric function* as

$$\mathcal{M}_\lambda(y_1, \ldots, y_m) = \sum_{\substack{i_1, \ldots, i_l \\ \text{distinct terms}}} y_{i_1}^{\lambda_1} \cdots y_{i_l}^{\lambda_l} = y_1^{\lambda_1} \cdots y_l^{\lambda_l} + \text{symmetric terms.}$$

Example 3. Let Y be an $m \times m$ symmetric matrix with eigenvalues $y_1, \ldots, y_m$:

(1)
$$M_{(1)}(Y) = y_1 + \cdots + y_m;$$

(2)
$$M_{(2)}(Y) = y_1^2 + \cdots + y_m^2;$$

(3)
$$\mathcal{M}_{(1,1)}(Y) = \sum_{i<j} y_i y_j;$$

(4)
$$M_{(2,1)}(Y) = \sum_{i,j} y_i^2 y_j.$$

Here we see the last two functions have the same length of partitions but different sum ranges, due to the fact that y_1y_2 and y_2y_1 are the same while $y_1^2y_2$ and $y_2^2y_1$ are different.

Remark 4. An explicit expression of $M_\lambda(Y)$ is given by

$$\mathcal{M}_{(1^{m_1}2^{m_2}\cdots)}(Y) = \left(\prod_{i=1}^{h} \frac{1}{m_i!}\right) \sum_{i_1,\ldots i_l} y_{i_1}^{\lambda_1} \cdots y_{i_l}^{\lambda_l}. \tag{1.2}$$

Theorem 5. *There exist some constants $c_{\kappa,\lambda}$, such that* [1, eq. 13, p. 234]

$$C_\kappa(Y) = \sum_{\lambda\le\kappa} c_{\kappa,\lambda}\mathcal{M}_\lambda(Y), \tag{1.3}$$

More over, the constant $c_{\kappa,\lambda}$ satisfies the recurrence [1, eq. 14, p. 234]

$$c_{\kappa,\lambda} = \sum_{\lambda<\mu\le\kappa} \frac{(\lambda_i+t)-(\lambda_j-t)}{\rho_\kappa-\rho_\lambda} c_{\kappa,\mu}, \tag{1.4}$$

where for $\lambda = (\lambda_1, \ldots, \lambda_l)$, the sum is over all $\mu = (\lambda_1, \ldots, \lambda_i + t, \ldots, \lambda_j - t, \ldots, \lambda_l)$ for $t = 1, \ldots, \lambda_j$ such that by rearranging tuple μ in a descending order, it lies as $\lambda < \mu \le \kappa$. The initial values are given by $c_{(n),(n)} = 1$ and for any $\lambda = (\lambda_1, \ldots, \lambda_l) \in \mathcal{P}_n$,

$$\sum_{\mu=\lambda}^{(n)} c_{\kappa,\mu} = \binom{n}{\lambda_1, \ldots, \lambda_l}, \tag{1.5}$$

which is derived from (1.1).

2. Package

MZonal(λ, $\vec{Y}$)

Compute specific expression $M_\lambda(\vec{Y})$ for partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ and variables $\vec{Y} = (y_1, \ldots, y_m)$, by (1.2).

EXAMPLE. Several computations. Note that when $m < l$, $M_\lambda(\vec{Y}) = 0$.

```
sage:  load('Zonal.sage')
sage:  var('a','b','c')
(a,b,c)
sage:  MZonal([2,2,1],[a,b,c])
a^2*b^2*c + a^2*b*c^2 + a*b^2*c^2
sage:  MZonal([2,1],[a,b,c])
a^2*b + a*b^2 + a^2*c + b^2*c + a*c^2 + b*c^2
sage:  MZonal([2,1,1,1],[a,b,c])
0
```

Coeffi(kappa,lambda)

Compute specific expression $c_{\kappa,\lambda}$ for partitions $\kappa = (\kappa_1, \ldots, \kappa_k) \ge \lambda = (\lambda_1, \ldots, \lambda_l)$, by (1.4) and (1.5).

EXAMPLE. Several computations. Note that if $\lambda > \kappa$, $c_{\kappa,\lambda} = 0$.

```
sage:  load('Zonal.sage')
sage:  Coeffi([5,4],[3,3,3])
82944/1925
sage:  Coeffi([5,4],[5,4])
768/11
sage:  Coeffi([9],[9])
1
sage:  Coeffi([3,3,3],[5,4])
0
```

CZonal(λ, $\vec{Y}$)

Compute specific expression $\mathcal{C}_\lambda(\vec{Y})$ for partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ and variables $\vec{Y} = (y_1, \ldots, y_m)$, by (1.3).

EXAMPLE. Several computations. Note that when $m < l$, $\mathcal{C}_\lambda(\vec{Y}) = 0$.

```
sage:  load('Zonal.sage')
sage:  var('a','b','c')
(a,b,c)
sage:  CZonal([2,2,1],[a,b,c])
32/3*a^2*b^2*c + 32/3*a^2*b*c^2 + 32/3*a*b^2*c^2
sage:  CZonal([2,1],[a,b,c])
12/5*a^2*b + 12/5*a*b^2 + 12/5*a^2*c + 18/5*a*b*c + 12/5*b^2*c +
12/5*a*c^2 + 12/5*b*c^2
sage:  CZonal([2,1,1,1],[a,b,c])
0
```

References

[1] R. Muirhead, *Aspects of Multivariate Statistical Theory*. John Wiley & Sons Inc., 1982.