# Implementation of an Algorithm on Converting Sums into Nested Sums

Lin Jiu

Tulane University & Research Institute for Symbolic Computation (RISC)

January 8, 2014

Joint Work with Carsten Schneider of RISC 

# Outlines

# Nested Sums

## Definition

An indefinite nested sum usually has the form

$$\begin{aligned} S(\nu) &= \sum_{\Lambda(\nu)\geq j_1\geq j_2\geq\cdots\geq j_n\geq 1} f_1(j_1)\, f_2(j_2)\ldots f_n(j_n)\,. \\ &= \sum_{j_1=1}^{\Lambda(\nu)} f_1(j_1) \sum_{j_2=1}^{j_1} f_2(j_2)\ldots f_{n-1}(j_{n-1}) \sum_{j_n=1}^{j_{n-1}} f_n(j_n) \end{aligned}$$

## Example

$$\sum_{j_1=0}^{m}\sum_{j_2=0}^{j_1}\cdots\sum_{j_n=0}^{j_{n-1}} 1 = \binom{m+n}{n}$$

# Nested Sums

## Definition

An indefinite nested sum usually has the form

$$S(\nu) = \sum_{\Lambda(\nu)\geq j_1\geq j_2\geq\cdots\geq j_n\geq 1} f_1(j_1) f_2(j_2)\ldots f_n(j_n).$$

$$= \sum_{j_1=1}^{\Lambda(\nu)} f_1(j_1) \sum_{j_2=1}^{j_1} f_2(j_2)\ldots f_{n-1}(j_{n-1}) \sum_{j_n=1}^{j_{n-1}} f_n(j_n)$$

## Example

$$\sum_{j_1=0}^{m}\sum_{j_2=0}^{j_1}\cdots\sum_{j_n=0}^{j_{n-1}} 1 = \binom{m+n}{n}$$

# Nested Sums Continued

## Examples

$$\begin{cases} Z(n; m_1, \dots m_k; x_1, \dots, x_k) := \sum_{n \geq i_1 > \cdots > i_k > 0} \frac{x_1^{i_1}}{i_1^{m_1}} \cdots \frac{x_k^{i_k}}{i_k^{m_k}} & \text{Z-sum} \\ S(n; m_1, \dots m_k; x_1, \dots, x_k) := \sum_{n \geq i_1 \geq \cdots \geq i_k \geq 1} \frac{x_1^{i_1}}{i_1^{m_1}} \cdots \frac{x_k^{i_k}}{i_k^{m_k}} & \text{S-sum} \\ H(n; a_1, \dots, a_k) := \sum_{n \geq i_1 \geq \cdots \geq i_k \geq 1} \frac{\text{sign}(a_1)^{i_1}}{i_1^{|a_1|}} \cdots \frac{\text{sign}(a_k)^{i_k}}{i_1^{|a_k|}} & \text{H-sum} \end{cases}$$

## Remark

$$Z(\infty; s; 1) = S(\infty; s; 1) = \sum_{i=1}^{\infty} \frac{1}{i^s} = \zeta(s)$$

$$H(n; 1) = \sum_{i=1}^{n} \frac{1}{i} = H_n;\ H(n; p(> o)) = \sum_{i=1}^{n} \frac{1}{i^p}$$

# Nested Sums Continued

## Examples

$$\begin{cases} Z(n; m_1, \ldots m_k; x_1, \ldots, x_k) := \sum_{n \geq i_1 > \cdots > i_k > 0} \frac{x_1^{i_1}}{i_1^{m_1}} \cdots \frac{x_k^{i_k}}{i_k^{m_k}} & \text{Z-sum} \\ S(n; m_1, \ldots m_k; x_1, \ldots, x_k) := \sum_{n \geq i_1 \geq \cdots \geq i_k \geq 1} \frac{x_1^{i_1}}{i_1^{m_1}} \cdots \frac{x_k^{i_k}}{i_k^{m_k}} & \text{S-sum} \\ H(n; a_1, \ldots, a_k) := \sum_{n \geq i_1 \geq \cdots \geq i_k \geq 1} \frac{\operatorname{sign}(a_1)^{i_1}}{i_1^{|a_1|}} \cdots \frac{\operatorname{sign}(a_k)^{i_k}}{i_1^{|a_k|}} & \text{H-sum} \end{cases}$$

## Remark

$$Z(\infty; s; 1) = S(\infty; s; 1) = \sum_{i=1}^{\infty} \frac{1}{i^s} = \zeta(s)$$

$$H(n; 1) = \sum_{i=1}^{n} \frac{1}{i} = H_n;\ H(n; p(> o)) = \sum_{i=1}^{n} \frac{1}{i^p}$$

# Why "Nested" Sums

$$S_1(\nu) := \sum_{x=1}^{\nu} \frac{1}{(x+\nu)^2}$$

$$S_2(\nu) := \sum_{x=0}^{\nu-1} \left[ \frac{1}{(1+2x)^2} + \underbrace{\frac{1}{(2+2x)^2} - \frac{1}{(1+x)^2}}_{=-\frac{3}{4(1+x)^2}} \right]$$

**Fact**

$$S_1 = S_2 =: S(\nu)$$

# Why "Nested" Sums

$$S_1(\nu) := \sum_{x=1}^{\nu} \frac{1}{(x+\nu)^2}$$

$$S_2(\nu) := \sum_{x=0}^{\nu-1} \left[ \frac{1}{(1+2x)^2} + \underbrace{\frac{1}{(2+2x)^2} - \frac{1}{(1+x)^2}}_{=-\frac{3}{4(1+x)^2}} \right]$$

**Fact**

$$S_1 = S_2 =: S(\nu)$$

# Why "Nested" Sums

$$S_1(\nu) := \sum_{x=1}^{\nu} \frac{1}{(x+\nu)^2}$$

$$S_2(\nu) := \sum_{x=0}^{\nu-1} \left[ \frac{1}{(1+2x)^2} + \underbrace{\frac{1}{(2+2x)^2} - \frac{1}{(1+x)^2}}_{=-\frac{3}{4(1+x)^2}} \right]$$

**Fact**

$$S_1 = S_2 =: S(\nu)$$

# Why "Nested" Sums

$$S_1(\nu) := \sum_{x=1}^{\nu} \frac{1}{(x+\nu)^2}$$

$$S_2(\nu) := \sum_{x=0}^{\nu-1} \left[ \frac{1}{(1+2x)^2} + \underbrace{\frac{1}{(2+2x)^2} - \frac{1}{(1+x)^2}}_{=-\frac{3}{4(1+x)^2}} \right]$$

**Fact**

$$S_1 = S_2 =: S(\nu)$$

# Why "Nested" Sums (Continued)

**[Question]What is the asymptotic behavior of $S(\nu)$?**

$$S_1(\nu) := \sum_{x=1}^{\nu} \frac{1}{(x+\nu)^2} \text{ [VS] } S_2(\nu) := \sum_{x=0}^{\nu-1} \left[\frac{1}{(1+2x)^2} - \frac{3}{4(1+x)^2}\right]$$

**[Anwser]**

$$S_2(\infty) = 0$$

**Reason I for Choosing Nested Sums**

**Asymptotics.**

# Why "Nested" Sums (Continued)

**[Question]What is the asymptotic behavior of $S(\nu)$?**

$$S_1(\nu) := \sum_{x=1}^{\nu} \frac{1}{(x+\nu)^2} \text{ [VS] } S_2(\nu) := \sum_{x=0}^{\nu-1} \left[\frac{1}{(1+2x)^2} - \frac{3}{4(1+x)^2}\right]$$

**[Anwser]**

$$S_2(\infty) = 0$$

**Reason I for Choosing Nested Sums**

**Asymptotics.**

# Why "Nested" Sums (Continued)

**[Question]What is the asymptotic behavior of $S(\nu)$?**

$$S_1(\nu) := \sum_{x=1}^{\nu} \frac{1}{(x+\nu)^2} \text{ [VS] } S_2(\nu) := \sum_{x=0}^{\nu-1} \left[\frac{1}{(1+2x)^2} - \frac{3}{4(1+x)^2}\right]$$

**[Anwser]**

$$S_2(\infty) = 0$$

**Reason I for Choosing Nested Sums**

**Asymptotics.**

# Why "Nested" Sums (Continued)

**[Question]What is the asymptotic behavior of $S(\nu)$?**

$$S_1(\nu) := \sum_{x=1}^{\nu} \frac{1}{(x+\nu)^2} \text{ [VS] } S_2(\nu) := \sum_{x=0}^{\nu-1} \left[ \frac{1}{(1+2x)^2} - \frac{3}{4(1+x)^2} \right]$$

**[Anwser]**

$$S_2(\infty) = 0$$

## Reason I for Choosing Nested Sums

**Asymptotics.**

# Why "Nested" Sums (Continued)

## Reason II for Choosing Nested Sums

**Convergence.**

## Example

$$S(\infty)=\sum_{i_1=1}^{\infty}\frac{\sum_{i_2=1}^{i_1}\frac{\sum_{i_3=1}^{i_2}\cdots}{i_2^{m_2}}}{i_1^{m_1}}.$$

## Theorem

*$S(\infty)$ is convergent iff $m_1>1$ [provided that $S(\infty)$ has only finitely many sums].*

# Why "Nested" Sums (Continued)

## Reason II for Choosing Nested Sums

**Convergence.**

## Example

$$S(\infty) = \sum_{i_1=1}^{\infty} \frac{\sum_{i_2=1}^{i_1} \frac{\sum_{i_3=1}^{i_2} \cdots}{i_2^{m_2}}}{i_1^{m_1}}.$$

## Theorem

*$S(\infty)$ is convergent iff $m_1 > 1$ [provided that $S(\infty)$ has only finitely many sums].*

# Why "Nested" Sums (Continued)

## Reason II for Choosing Nested Sums

**Convergence.**

## Example

$$S(\infty) = \sum_{i_1=1}^{\infty} \frac{\sum_{i_2=1}^{i_1} \frac{\sum_{i_3=1}^{i_2} \cdots}{i_2^{m_2}}}{i_1^{m_1}}.$$

## Theorem

*$S(\infty)$ is convergent iff $m_1 > 1$ [provided that $S(\infty)$ has only finitely many sums].*

# Why "Nested" Sums (Continued)

## Reason III for Choosing Nested Sums

**"Algebraic Relations to Reduce the Sum".**

## Remark

This is what the "Sigma.m" package does. [A mathematica package that Carsten works on for years to deal with "all"(at least hypergeometric) sums/products]

## Algebraic Relations

In short, given a nested sum, we want to express it in terms of known special functions, e.g. S-sums, Z-sums, ${}_AF_B$, etc. Algebraic relations could help us to find the minimal basis for the expression and reduce the sum.

# Why "Nested" Sums (Continued)

## Reason III for Choosing Nested Sums

**"Algebraic Relations to Reduce the Sum".**

## Remark

This is what the "Sigma.m" package does. [A mathematica package that Carsten works on for years to deal with "all"(at least hypergeometric) sums/products]

## Algebraic Relations

In short, given a nested sum, we want to express it in terms of known special functions, e.g. S-sums, Z-sums, ${}_AF_B$, etc. Algebraic relations could help us to find the minimal basis for the expression and reduce the sum.

# Why "Nested" Sums (Continued)

## Reason III for Choosing Nested Sums

**"Algebraic Relations to Reduce the Sum".**

## Remark

This is what the "Sigma.m" package does. [A mathematica package that Carsten works on for years to deal with "all"(at least hypergeometric) sums/products]

## Algebraic Relations

In short, given a nested sum, we want to express it in terms of known special functions, e.g. S-sums, Z-sums, ${}_AF_B$, etc. Algebraic relations could help us to find the minimal basis for the expression and reduce the sum.

# Sigma.m Package

"Sigma is a Mathematica package that can handle multisums in terms of indefinite nested sums and products. The summation principles of Sigma are: telescoping, creative telescoping and recurrence solving. The underlying machinery of Sigma is based on difference field theory. The package has been developed by Carsten Schneider, a member of the RISC Combinatorics group. "

"The source code for this package is password protected. To get the password send an email to Peter Paule. It will be given for free to all researchers and non-commercial users.



**http://www.risc.jku.at/research/combinat/software/Sigma/index.php**

# Sigma.m Package (Continued)

In short, “Sigma.m” deals with general multisums.

The algorithm I am working on deals with special case.

# Form of the Required Sums

$$S(\nu) = \sum_{i_1=a_1(\nu)}^{b_1(\nu)} \sum_{i_2=a_2(\nu,i_1)}^{b_2(\nu,i_1)} \cdots \sum_{i_n=a_n(\nu,i_1,\ldots i_{n-1})}^{b_n(\nu,i_1\ldots,i_{n-1})} \frac{\lambda_1^{i_1} \ldots \lambda_n^{i_n}}{\prod_r L_r(\nu, i_1, \ldots, i_n)^{p_r}},$$

where

$$\begin{cases} \lambda_k \in \mathbb{C} \\ p_k \in \mathbb{N} = \{1, 2, \ldots\} \\ L_r \text{ is a linear polynomial with integer coefficients} \\ a_k, b_k \text{ are either infinity or linear polynomials with integer coefficients} \end{cases}$$

NOTE: The number of products in the denominator is unknown.

# Form of the Required Sums

$$S(\nu) = \sum_{i_1=a_1(\nu)}^{b_1(\nu)} \sum_{i_2=a_2(\nu,i_1)}^{b_2(\nu,i_1)} \cdots \sum_{i_n=a_n(\nu,i_1,\ldots i_{n-1})}^{b_n(\nu,i_1\ldots,i_{n-1})} \frac{\lambda_1^{i_1}\ldots\lambda_n^{i_n}}{\prod_r L_r(\nu,i_1,\ldots,i_n)^{p_r}},$$

where

$$\begin{cases} \lambda_k \in \mathbb{C} \\ p_k \in \mathbb{N} = \{1,2,\ldots\} \\ L_r \text{ is a linear polynomial with integer coefficients} \\ a_k, b_k \text{ are either infinity or linear polynomials with integer coefficients} \end{cases}$$

NOTE: The number of products in the denominator is unknown.

# Examples

## Examples

$$\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(k+m)^2(2m+4k+1)}=4-2\mathcal{C}\pi-\frac{\pi^2}{6}-4\log 2+\frac{21}{4}\zeta(3).$$

$$\sum_{k=1}^{\infty}\sum_{m=1}^{k}\frac{(-1)^{k+m}}{(k+1)^2(2m+1)}=8\mathrm{Im}\left[Z\left(\infty;2,1;-i,i\right)\right]+4\mathcal{C}\log 2-\frac{\pi^3}{8}-\frac{\pi^2}{12}.$$

Where, $\mathcal{C}=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}$—Catalan Constant

# An Important Notation

## Notation

$$\sum_{k=a}^{b}{}' f(k) = \begin{cases} \sum_{k=a}^{b} f(k) & a \leq b \\ 0 & a = b+1 \\ -\sum_{k=b+1}^{a-1} f(k) & a \geq b+2 \end{cases}.$$

## Example

$$\sum_{x=5}^{2} x = -\sum_{x=3}^{4} x = -7$$

# An Important Notation

## Notation

$$\sum_{k=a}^{b}{}' f(k) = \begin{cases} \sum_{k=a}^{b} f(k) & a \leq b \\ 0 & a = b+1 \\ -\sum_{k=b+1}^{a-1} f(k) & a \geq b+2 \end{cases}.$$

## Example

$$\sum_{x=5}^{2} x = -\sum_{x=3}^{4} x = -7$$

# Key Idea——Invariant under Shifting

Find a set of integers $(\Delta_0, \Delta_1, \ldots, \Delta_n)$, s.t.
(1) $\Delta_0 \neq 0$
(2) $\forall r$,

$$L_r(\nu + \Delta_0, i_1 + \Delta_1, \ldots, i_n + \Delta_n) = L_r(\nu, i_1, \ldots, i_n).$$

**[Question] Does such an integer vector always exist?**
**[Answer] No. For example, when the number of $L_r$'s is larger than the number of sums.**

Example

$$S(\nu) = \sum_{m=0}^{\infty} \frac{1}{(\nu + m)^2 (4\nu + 2m + 1)}.$$

# Key Idea——Invariant under Shifting

Find a set of integers $(\Delta_0, \Delta_1, \ldots, \Delta_n)$, s.t.
(1) $\Delta_0 \neq 0$
(2) $\forall r$,

$$L_r(\nu + \Delta_0, i_1 + \Delta_1, \ldots, i_n + \Delta_n) = L_r(\nu, i_1, \ldots, i_n).$$

**[Question] Does such an integer vector always exist?**

**[Answer] No. For example, when the number of $L_r$'s is larger than the number of sums.**

Example

$$S(\nu) = \sum_{m=0}^{\infty} \frac{1}{(\nu+m)^2 (4\nu + 2m + 1)}.$$

# Key Idea——Invariant under Shifting

Find a set of integers $(\Delta_0, \Delta_1, \ldots, \Delta_n)$, s.t.
(1) $\Delta_0 \neq 0$
(2) $\forall r$,

$$L_r (\nu + \Delta_0, i_1 + \Delta_1, \ldots, i_n + \Delta_n) = L_r (\nu, i_1, \ldots, i_n).$$

**[Question] Does such an integer vector always exist?**
**[Answer] No. For example, when the number of $L_r$'s is larger than the number of sums.**

## Example

$$S(\nu) = \sum_{m=0}^{\infty} \frac{1}{(\nu + m)^2 (4\nu + 2m + 1)}.$$

# Step I——Partial Fraction Decomposition(PFD)

## Step I——Partial Fraction Decomposition(PFD)

**KEY: From the Innermost Sum Index to the Outermost**

## Example

$$\sum_{x\geq y\geq z}\frac{1}{(x+y)(y+z)(x+z)}=\sum_{x\geq y\geq z}\frac{1}{2x}\left[\frac{1}{x+y}+\frac{1}{x-y}\right]\left[\frac{1}{y+z}-\frac{1}{x+z}\right].$$

## Result

**For each part of the sum, the number of $L_r$'s is NO GREATER THAN the number of sums, i.e. it guarantees the existence of the invariant shifting vector $(\Delta_0, \Delta_1, \ldots, \Delta_n)$.**

## NOTE

**The solution $(\Delta_0, \Delta_1, \ldots, \Delta_n)$ has at least one free variable to turn rational solutions into integer solutions.**

# Step I——Partial Fraction Decomposition(PFD)

## Step I——Partial Fraction Decomposition(PFD)

**KEY: From the Innermost Sum Index to the Outermost**

## Example

$$\sum_{x \geq y \geq z} \frac{1}{(x+y)(y+z)(x+z)} = \sum_{x \geq y \geq z} \frac{1}{2x} \left[\frac{1}{x+y} + \frac{1}{x-y}\right] \left[\frac{1}{y+z} - \frac{1}{x+z}\right].$$

## Result

**For each part of the sum, the number of $L_r$'s is NO GREATER THAN the number of sums, i.e. it guarantees the existence of the invariant shifting vector $(\Delta_0, \Delta_1, \ldots, \Delta_n)$.**

## NOTE

**The solution $(\Delta_0, \Delta_1, \ldots, \Delta_n)$ has at least one free variable to turn rational solutions into integer solutions.**

# Step I——Partial Fraction Decomposition(PFD)

## Step I——Partial Fraction Decomposition(PFD)

**KEY: From the Innermost Sum Index to the Outermost**

## Example

$$\sum_{x\geq y\geq z}\frac{1}{(x+y)(y+z)(x+z)}=\sum_{x\geq y\geq z}\frac{1}{2x}\left[\frac{1}{x+y}+\frac{1}{x-y}\right]\left[\frac{1}{y+z}-\frac{1}{x+z}\right].$$

## Result

**For each part of the sum, the number of $L_r$'s is NO GREATER THAN the number of sums, i.e. it guarantees the existence of the invariant shifting vector $(\Delta_0,\Delta_1,\ldots,\Delta_n)$.**

## NOTE

**The solution $(\Delta_0,\Delta_1,\ldots,\Delta_n)$ has at least one free variable to turn rational solutions into integer solutions.**

# Step I——Partial Fraction Decomposition(PFD)

Step I——Partial Fraction Decomposition(PFD)

**KEY: From the Innermost Sum Index to the Outermost**

Example

$$\sum_{x\geq y\geq z}\frac{1}{(x+y)(y+z)(x+z)}=\sum_{x\geq y\geq z}\frac{1}{2x}\left[\frac{1}{x+y}+\frac{1}{x-y}\right]\left[\frac{1}{y+z}-\frac{1}{x+z}\right].$$

Result

**For each part of the sum, the number of $L_r$'s is NO GREATER THAN the number of sums, i.e. it guarantees the existence of the invariant shifting vector $(\Delta_0,\Delta_1,\ldots,\Delta_n)$.**

NOTE

**The solution $(\Delta_0,\Delta_1,\ldots,\Delta_n)$ has at least one free variable to turn rational solutions into integer solutions.**

# Step II——Key Step (Idea)

For a (not completely known) function $f(x)$, suppose we want to compute $f(101)$, by knowing the following

$$\begin{cases} f(2) \\ \Delta f(x) := f(x+3) - f(x) \end{cases}.$$

Noting that

$$101 \equiv 2 \ (mod\, 3)$$

$$\begin{aligned} f(101) &= f(98) + [f(101) - f(98)] \\ &= f(98) + \Delta f(98) \\ &= \ldots \\ &= f(2) + \sum_{k=0}^{32} \Delta f(3k+2). \end{aligned}$$

## Step II——Key Step (Idea)

For a (not completely known) function $f(x)$, suppose we want to compute $f(101)$, by knowing the following

$$\begin{cases} f(2) \\ \Delta f(x) := f(x+3) - f(x) \end{cases}.$$

Noting that

$$101 \equiv 2 \ (mod\, 3)$$

$$\begin{aligned} f(101) &= f(98) + [f(101) - f(98)] \\ &= f(98) + \Delta f(98) \\ &= \dots \\ &= f(2) + \sum_{k=0}^{32} \Delta f(3k+2). \end{aligned}$$

# Step II——Key Step (Idea)

For a (not completely known) function $f(x)$, suppose we want to compute $f(101)$, by knowing the following

$$\begin{cases} f(2) \\ \Delta f(x) := f(x+3) - f(x) \end{cases}.$$

Noting that

$$101 \equiv 2 \ (mod\, 3)$$

$$\begin{aligned} f(101) &= f(98) + [f(101) - f(98)] \\ &= f(98) + \Delta f(98) \\ &= \dots \\ &= f(2) + \sum_{k=0}^{32} \Delta f(3k+2). \end{aligned}$$

# Step II——Key Step (Idea)

For a (not completely known) function $f(x)$, suppose we want to compute $f(101)$, by knowing the following

$$\begin{cases} f(2) \\ \Delta f(x) := f(x+3) - f(x) \end{cases}.$$

Noting that

$$101 \equiv 2 \ (mod\, 3)$$

$$\begin{aligned} f(101) &= f(98) + [f(101) - f(98)] \\ &= f(98) + \Delta f(98) \\ &= \dots \\ &= f(2) + \sum_{k=0}^{32} \Delta f(3k+2). \end{aligned}$$

# Step II——Key Step (Idea)

For a (not completely known) function $f(x)$, suppose we want to compute $f(101)$, by knowing the following

$$\begin{cases} f(2) \\ \Delta f(x) := f(x+3) - f(x) \end{cases}.$$

Noting that

$$101 \equiv 2 \ (mod\ 3)$$

$$\begin{aligned} f(101) &= f(98) + [f(101) - f(98)] \\ &= f(98) + \Delta f(98) \\ &= \ldots \\ &= f(2) + \sum_{k=0}^{32} \Delta f(3k+2). \end{aligned}$$

# Step II——Key Step (by Induction)

Now, we could consider

$$S(\nu) = \sum_{i_1=a_1(\nu)}^{b_1(\nu)} \sum_{i_2=a_2(\nu,i_1)}^{b_2(\nu,i_1)} \cdots \sum_{i_n=a_n(\nu,i_1,\ldots i_{n-1})}^{b_n(\nu,i_1\ldots,i_{n-1})} \frac{\lambda_1^{i_1}\ldots\lambda_n^{i_n}}{\prod_{r=1}^{n} L_r(\nu, i_1, \ldots, i_n)^{p_r}},$$

with

$$L_r(\nu + \Delta_0, i_1 + \Delta_1, \ldots, i_n + \Delta_n) = L_r(\nu, i_1, \ldots, i_n), r = 1, \ldots, n$$

Define

$$\lambda := \prod_{k=1}^{n} \lambda_k^{\Delta_k},$$

and

$$\Delta S(\nu) := S(\nu + \Delta_0) - \lambda S(\nu).$$

# Step II——Key Step (by Induction)

Now, we could consider

$$S(\nu) = \sum_{i_1=a_1(\nu)}^{b_1(\nu)} \sum_{i_2=a_2(\nu,i_1)}^{b_2(\nu,i_1)} \cdots \sum_{i_n=a_n(\nu,i_1,\dots i_{n-1})}^{b_n(\nu,i_1\dots,i_{n-1})} \frac{\lambda_1^{i_1}\dots\lambda_n^{i_n}}{\prod_{r=1}^{n} L_r(\nu, i_1,\dots,i_n)^{p_r}},$$

with

$$L_r(\nu+\Delta_0, i_1+\Delta_1,\dots,i_n+\Delta_n) = L_r(\nu, i_1,\dots,i_n), r=1,\dots,n$$

Define

$$\lambda := \prod_{k=1}^{n} \lambda_k^{\Delta_k},$$

and

$$\Delta S(\nu) := S(\nu+\Delta_0) - \lambda S(\nu).$$

# Step II——Key Step (by Induction)

Now, we could consider

$$S(\nu) = \sum_{i_1=a_1(\nu)}^{b_1(\nu)} \sum_{i_2=a_2(\nu,i_1)}^{b_2(\nu,i_1)} \cdots \sum_{i_n=a_n(\nu,i_1,\ldots i_{n-1})}^{b_n(\nu,i_1\ldots,i_{n-1})} \frac{\lambda_1^{i_1}\ldots\lambda_n^{i_n}}{\prod_{r=1}^{n} L_r(\nu,i_1,\ldots,i_n)^{p_r}},$$

with

$$L_r(\nu+\Delta_0, i_1+\Delta_1,\ldots,i_n+\Delta_n) = L_r(\nu,i_1,\ldots,i_n), r=1,\ldots,n$$

Define

$$\lambda := \prod_{k=1}^{n} \lambda_k^{\Delta_k},$$

and

$$\Delta S(\nu) := S(\nu+\Delta_0) - \lambda S(\nu).$$

# Step II——Key Step (by Induction)

Now, we could consider

$$S(\nu)=\sum_{i_1=a_1(\nu)}^{b_1(\nu)}\sum_{i_2=a_2(\nu,i_1)}^{b_2(\nu,i_1)}\cdots\sum_{i_n=a_n(\nu,i_1,\ldots i_{n-1})}^{b_n(\nu,i_1\ldots,i_{n-1})}\frac{\lambda_1^{i_1}\ldots\lambda_n^{i_n}}{\prod_{r=1}^{n}L_r(\nu,i_1,\ldots,i_n)^{p_r}},$$

with

$$L_r(\nu+\Delta_0,i_1+\Delta_1,\ldots,i_n+\Delta_n)=L_r(\nu,i_1,\ldots,i_n),\, r=1,\ldots,n$$

Define

$$\lambda:=\prod_{k=1}^{n}\lambda_k^{\Delta_k},$$

and

$$\Delta S(\nu):=S(\nu+\Delta_0)-\lambda S(\nu).$$

# Step II——Key Step(Continued)

Define

$$\begin{cases} \alpha_k := & a_k(\nu+\Delta_0, i_1+\Delta_1, \ldots, i_{k-1}+\Delta_{k-1}) - a_k(\nu, i_1, \ldots, i_{k-1}) \in \mathbb{Z} \\ \beta_k := & b_k(\nu+\Delta_0, i_1+\Delta_1, \ldots, i_{k-1}+\Delta_{k-1}) - b_k(\nu, i_1, \ldots, i_{k-1}) \in \mathbb{Z} \end{cases}.$$

Then

$$\begin{aligned} S(\nu+\Delta_0) &= \sum_{i_1=a_1(\nu+\Delta_0)}^{b_1(\nu+\Delta_0)} \cdots \sum_{i_n=a_n(\nu+\Delta_0,i_1,\ldots i_{n-1})}^{b_n(\nu+\Delta_0,i_1\ldots,i_{n-1})} \frac{\lambda_1^{i_1}\ldots\lambda_n^{i_n}}{\prod_{r=1}^{n} L_r(\nu+\Delta_0, i_1, \ldots, i_n)^{p_r}} \\ [i_k \mapsto i_k+\Delta_k] &= \sum_{i_1+\Delta_1=a_1+\alpha_1}^{b_1+\beta_1} \cdots \sum_{i_n+\Delta_n=a_n+\alpha_n}^{b_n+\beta_n} \frac{\lambda_1^{i_1+\Delta_1}\ldots\lambda_n^{i_n+\Delta_n}}{\prod_{r=1}^{n} L_r(\nu+\Delta_0, i_1+\Delta_1, \ldots, i_n+\Delta_n)^{p_r}} \\ &= \lambda \sum_{i_1=a_1+\alpha_1-\Delta_1}^{b_1+\beta_1-\Delta_1} \cdots \sum_{i_n=a_n+\alpha_n-\Delta_n}^{b_n+\beta_n-\Delta_n} \underbrace{\frac{\lambda_1^{i_1}\ldots\lambda_n^{i_n}}{\prod_{r=1}^{n} L_r(\nu+\Delta_0, i_1, \ldots, i_n)^{p_r}}}_{su:=}. \end{aligned}$$

# Step II——Key Step(Continued)

Define

$$\begin{cases} \alpha_k := & a_k\left(\nu+\Delta_0, i_1+\Delta_1, \ldots, i_{k-1}+\Delta_{k-1}\right) - a_k\left(\nu, i_1, \ldots, i_{k-1}\right) \in \mathbb{Z} \\ \beta_k := & b_k\left(\nu+\Delta_0, i_1+\Delta_1, \ldots, i_{k-1}+\Delta_{k-1}\right) - b_k\left(\nu, i_1, \ldots, i_{k-1}\right) \in \mathbb{Z} \end{cases}.$$

Then

$$\begin{aligned} S\left(\nu+\Delta_0\right) &= \sum_{i_1=a_1(\nu+\Delta_0)}^{b_1(\nu+\Delta_0)} \cdots \sum_{i_n=a_n\left(\nu+\Delta_0, i_1, \ldots i_{n-1}\right)}^{b_n\left(\nu+\Delta_0, i_1 \ldots, i_{n-1}\right)} \frac{\lambda_1^{i_1} \ldots \lambda_n^{i_n}}{\prod_{r=1}^{n} L_r\left(\nu+\Delta_0, i_1, \ldots, i_n\right)^{p_r}} \\ [i_k \mapsto i_k+\Delta_k] &= \sum_{i_1+\Delta_1=a_1+\alpha_1}^{b_1+\beta_1} \cdots \sum_{i_n+\Delta_n=a_n+\alpha_n}^{b_n+\beta_n} \frac{\lambda_1^{i_1+\Delta_1} \ldots \lambda_n^{i_n+\Delta_n}}{\prod_{r=1}^{n} L_r\left(\nu+\Delta_0, i_1+\Delta_1, \ldots, i_n+\Delta_n\right)^{p_r}} \\ &= \lambda \sum_{i_1=a_1+\alpha_1-\Delta_1}^{b_1+\beta_1-\Delta_1} \cdots \sum_{i_n=a_n+\alpha_n-\Delta_n}^{b_n+\beta_n-\Delta_n} \underbrace{\frac{\lambda_1^{i_1} \ldots \lambda_n^{i_n}}{\prod_{r=1}^{n} L_r\left(\nu+\Delta_0, i_1, \ldots, i_n\right)^{p_r}}}_{su:=}. \end{aligned}$$

# Step II——Key Step(Continued)

$$
\begin{aligned}
\Delta S(\nu) &= S(\nu+\Delta_0) - \lambda S(\nu) \\
&= \lambda \left[ \sum_{i_1=a_1+\alpha_1-\Delta_1}^{b_1+\beta_1-\Delta_1} \cdots \sum_{i_n=a_n+\alpha_n-\Delta_n}^{b_n+\beta_n-\Delta_n} - \sum_{i_1=a_1}^{b_1} \cdots \sum_{i_n=a_n}^{b_n} \right] su
\end{aligned}
$$

$$
\begin{aligned}
& \sum_{i_1=a_1+\alpha_1-\Delta_1}^{b_1+\beta_1-\Delta_1} \cdots \sum_{i_n=a_n+\alpha_n-\Delta_n}^{b_n+\beta_n-\Delta_n} - \sum_{i_1=a_1}^{b_1} \cdots \sum_{i_n=a_n}^{b_n} \\
= & \left( \sum_{i_1=a_1+\alpha_1-\Delta_1}^{a_1-1} + \sum_{i_1=a_1}^{b_1} + \sum_{i_1=b_1+1}^{b_1+\beta_1-\Delta_1} \right) \cdots \left( \sum_{i_n=a_n+\alpha_n-\Delta_n}^{a_n-1} + \sum_{i_n=a_n}^{b_n} + \sum_{i_n=b_n+1}^{b_n+\beta_n-\Delta_n} \right) - \sum_{i_1=a_1}^{b_1} \cdots \sum_{i_n=a_n}^{b_n} \\
= & \left( \underbrace{\sum_{i_1=a_1+\alpha_1-\Delta_1}^{a_1-1} + \sum_{i_1=b_1+1}^{b_1+\beta_1-\Delta_1}}_{\text{Summable}} + \underbrace{\sum_{i_1=a_1}^{b_1}}_{\text{Symbolic}} \right) \cdots \left( \underbrace{\sum_{i_n=a_n+\alpha_n-\Delta_n}^{a_n-1} + \sum_{i_n=b_n+1}^{b_n+\beta_n-\Delta_n}}_{\text{Summable}} + \underbrace{\sum_{i_n=a_n}^{b_n}}_{\text{Symbolic}} \right) - \sum_{i_1=a_1}^{b_1} \cdots \sum_{i_n=a_n}^{b_n}
\end{aligned}
$$

**$\Delta S(\nu)$ has at least one quantifier less. Thus, by induction, it can be converted into nested sum**

# Step II——Key Step(Continued)

$$\Delta S(\nu) = S(\nu+\Delta_0) - \lambda S(\nu) = \lambda\left[\sum_{i_1=a_1+\alpha_1-\Delta_1}^{b_1+\beta_1-\Delta_1}\cdots\sum_{i_n=a_n+\alpha_n-\Delta_n}^{b_n+\beta_n-\Delta_n} - \sum_{i_1=a_1}^{b_1}\cdots\sum_{i_n=a_n}^{b_n}\right] su$$

$$\sum_{i_1=a_1+\alpha_1-\Delta_1}^{b_1+\beta_1-\Delta_1}\cdots\sum_{i_n=a_n+\alpha_n-\Delta_n}^{b_n+\beta_n-\Delta_n} - \sum_{i_1=a_1}^{b_1}\cdots\sum_{i_n=a_n}^{b_n}$$

$$= \left(\sum_{i_1=a_1+\alpha_1-\Delta_1}^{a_1-1} + \sum_{i_1=a_1}^{b_1} + \sum_{i_1=b_1+1}^{b_1+\beta_1-\Delta_1}\right)\cdots\left(\sum_{i_n=a_n+\alpha_n-\Delta_n}^{a_n-1} + \sum_{i_n=a_n}^{b_n} + \sum_{i_n=b_n+1}^{b_n+\beta_n-\Delta_n}\right) - \sum_{i_1=a_1}^{b_1}\cdots\sum_{i_n=a_n}^{b_n}$$

$$= \left(\underbrace{\sum_{i_1=a_1+\alpha_1-\Delta_1}^{a_1-1} + \sum_{i_1=b_1+1}^{b_1+\beta_1-\Delta_1}}_{\text{Summable}} + \underbrace{\sum_{i_1=a_1}^{b_1}}_{\text{Symbolic}}\right)\cdots\left(\underbrace{\sum_{i_n=a_n+\alpha_n-\Delta_n}^{a_n-1} + \sum_{i_n=b_n+1}^{b_n+\beta_n-\Delta_n}}_{\text{Summable}} + \underbrace{\sum_{i_n=a_n}^{b_n}}_{\text{Symbolic}}\right) - \sum_{i_1=a_1}^{b_1}\cdots\sum_{i_n=a_n}^{b_n}$$

**$\Delta S(\nu)$ has at least one quantifier less. Thus, by induction, it can be converted into nested sum**

# Step II——Key Step(Continued)

$$\begin{aligned}\Delta S(\nu) &= S(\nu+\Delta_0) - \lambda S(\nu) \\ &= \lambda\left[\sum_{i_1=a_1+\alpha_1-\Delta_1}^{b_1+\beta_1-\Delta_1}\cdots\sum_{i_n=a_n+\alpha_n-\Delta_n}^{b_n+\beta_n-\Delta_n} - \sum_{i_1=a_1}^{b_1}\cdots\sum_{i_n=a_n}^{b_n}\right] su\end{aligned}$$

$$\begin{aligned}&\sum_{i_1=a_1+\alpha_1-\Delta_1}^{b_1+\beta_1-\Delta_1}\cdots\sum_{i_n=a_n+\alpha_n-\Delta_n}^{b_n+\beta_n-\Delta_n} - \sum_{i_1=a_1}^{b_1}\cdots\sum_{i_n=a_n}^{b_n} \\ =&\left(\sum_{i_1=a_1+\alpha_1-\Delta_1}^{a_1-1} + \sum_{i_1=a_1}^{b_1} + \sum_{i_1=b_1+1}^{b_1+\beta_1-\Delta_1}\right)\cdots\left(\sum_{i_n=a_n+\alpha_n-\Delta_n}^{a_n-1} + \sum_{i_n=a_n}^{b_n} + \sum_{i_n=b_n+1}^{b_n+\beta_n-\Delta_n}\right) - \sum_{i_1=a_1}^{b_1}\cdots\sum_{i_n=a_n}^{b_n} \\ =&\left(\underbrace{\sum_{i_1=a_1+\alpha_1-\Delta_1}^{a_1-1} + \sum_{i_1=b_1+1}^{b_1+\beta_1-\Delta_1}}_{\text{Summable}} + \underbrace{\sum_{i_1=a_1}^{b_1}}_{\text{Symbolic}}\right)\cdots\left(\underbrace{\sum_{i_n=a_n+\alpha_n-\Delta_n}^{a_n-1} + \sum_{i_n=b_n+1}^{b_n+\beta_n-\Delta_n}}_{\text{Summable}} + \underbrace{\sum_{i_n=a_n}^{b_n}}_{\text{Symbolic}}\right) - \sum_{i_1=a_1}^{b_1}\cdots\sum_{i_n=a_n}^{b_n}\end{aligned}$$

**$\Delta S(\nu)$ has at least one quantifier less. Thus, by induction, it can be converted into nested sum**

# Step II——Key Step(Continued)

Recall:

$$f(101) = f(2) + \sum_{k=0}^{32} \Delta f(3k+2).$$

Define

$$\text{Proj}(m,n,d) = \frac{1}{d}\sum_{k=1}^{d} \exp\left(2\pi i k \frac{m-n}{d}\right) = \begin{cases} 0 & m \not\equiv n \ (mod\ d) \\ 1 & m \equiv n \ (mod\ d) \end{cases}.$$

Then,

$$f(101) = f(2) + \sum_{k=2}^{98(=101-3)} \text{Proj}(k,2,3)\,\Delta f(k).$$

# Step II——Key Step(Continued)

Recall:

$$f(101) = f(2) + \sum_{k=0}^{32} \Delta f(3k+2).$$

Define

$$\mathrm{Proj}(m,n,d) = \frac{1}{d}\sum_{k=1}^{d} \exp\left(2\pi i k \frac{m-n}{d}\right) = \begin{cases} 0 & m \not\equiv n \ (mod\ d) \\ 1 & m \equiv n \ (mod\ d) \end{cases}.$$

Then,

$$f(101) = f(2) + \sum_{k=2}^{98(=101-3)} \mathrm{Proj}(k,2,3)\,\Delta f(k).$$

# Step II——Key Step(Continued)

Recall:

$$f(101) = f(2) + \sum_{k=0}^{32} \Delta f(3k+2).$$

Define

$$\mathrm{Proj}(m,n,d) = \frac{1}{d}\sum_{k=1}^{d} \exp\left(2\pi i k \frac{m-n}{d}\right) = \begin{cases} 0 & m \not\equiv n \ (mod\ d) \\ 1 & m \equiv n \ (mod\ d) \end{cases}.$$

Then,

$$f(101) = f(2) + \sum_{k=2}^{98(=101-3)} \mathrm{Proj}(k,2,3)\,\Delta f(k).$$

# Step II——Key Step(Continued)

Recall:

$$f(101) = f(2) + \sum_{k=0}^{32} \Delta f(3k+2).$$

Define

$$\text{Proj}(m,n,d) = \frac{1}{d}\sum_{k=1}^{d} \exp\left(2\pi i k \frac{m-n}{d}\right) = \begin{cases} 0 & m \not\equiv n \ (mod\ d) \\ 1 & m \equiv n \ (mod\ d) \end{cases}.$$

Then,

$$f(101) = f(2) + \sum_{k=2}^{98(=101-3)} \text{Proj}(k,2,3)\,\Delta f(k).$$

# Step II——Key Step(Continued)

Suppose $\nu_0 \in \{0, 1, \ldots, \Delta_0\}$, s.t. $\nu \equiv \nu_0 \ (mod\, \Delta_0)$, then

$$S(\nu) = \lambda^{\frac{\nu-\nu_0}{\Delta_0}} S(\nu_0) + \sum_{\mu=\nu_0}^{\nu-\Delta_0} \mathrm{Proj}(\mu, \nu_0, \Delta_0) \Delta S(\nu) \lambda^{\frac{\nu-\mu-\Delta_0}{\Delta_0}}.$$

**[Question] How to deal with $S(\nu_0)$?**

**FACT(Good News)**

$$\nu_0 \in \{0, 1, \ldots, \Delta_0\} \Longrightarrow S(\nu_0) \text{ is determined.}$$

**[Solution] Rewrite the sum.**

# Step II——Key Step(Continued)

Suppose $\nu_0 \in \{0, 1, \ldots, \Delta_0\}$, s.t. $\nu \equiv \nu_0 \ (mod \, \Delta_0)$, then

$$S(\nu) = \lambda^{\frac{\nu-\nu_0}{\Delta_0}} S(\nu_0) + \sum_{\mu=\nu_0}^{\nu-\Delta_0} \mathrm{Proj}(\mu, \nu_0, \Delta_0) \Delta S(\nu) \lambda^{\frac{\nu-\mu-\Delta_0}{\Delta_0}}.$$

**[Question] How to deal with $S(\nu_0)$?**

FACT(Good News)

$$\nu_0 \in \{0, 1, \ldots, \Delta_0\} \Longrightarrow S(\nu_0) \text{ is determined.}$$

**[Solution] Rewrite the sum.**

# Step II——Key Step(Continued)

Suppose $\nu_0 \in \{0, 1, \ldots, \Delta_0\}$, s.t. $\nu \equiv \nu_0 \ (mod\, \Delta_0)$, then

$$S(\nu) = \lambda^{\frac{\nu-\nu_0}{\Delta_0}} S(\nu_0) + \sum_{\mu=\nu_0}^{\nu-\Delta_0} \mathrm{Proj}(\mu, \nu_0, \Delta_0) \Delta S(\nu) \lambda^{\frac{\nu-\mu-\Delta_0}{\Delta_0}}.$$

**[Question] How to deal with $S(\nu_0)$?**

**FACT(Good News)**

$$\nu_0 \in \{0, 1, \ldots, \Delta_0\} \Longrightarrow S(\nu_0) \text{ is determined.}$$

**[Solution] Rewrite the sum.**

# Step II——Key Step(Continued)

Suppose $\nu_0 \in \{0, 1, \ldots, \Delta_0\}$, s.t. $\nu \equiv \nu_0 \ (mod\,\Delta_0)$, then

$$S(\nu) = \lambda^{\frac{\nu-\nu_0}{\Delta_0}} S(\nu_0) + \sum_{\mu=\nu_0}^{\nu-\Delta_0} \mathrm{Proj}(\mu, \nu_0, \Delta_0)\, \Delta S(\nu)\, \lambda^{\frac{\nu-\mu-\Delta_0}{\Delta_0}}.$$

**[Question] How to deal with $S(\nu_0)$?**

**FACT(Good News)**

$$\nu_0 \in \{0, 1, \ldots, \Delta_0\} \Longrightarrow S(\nu_0) \text{ is determined.}$$

**[Solution] Rewrite the sum.**

# Step II——Key Step(Continued)

$$S(\nu) = \lambda^{\frac{\nu-\nu_0}{\Delta_0}} S(\nu_0) + \sum_{\mu=\nu_0}^{\nu-\Delta_0} \mathrm{Proj}(\mu, \nu_0, \Delta_0) \Delta S(\mu) \lambda^{\frac{\nu-\mu-\Delta_0}{\Delta_0}}$$

**Replacing $\nu_0$ by $\mu_0$:**

$$S(\nu) = \lambda^{\frac{\nu-\mu_0}{\Delta_0}} S(\mu_0) + \sum_{\mu=\mu_0}^{\nu-\Delta_0} \mathbf{Proj}(\mu, \mu_0, \Delta_0) \Delta S(\mu) \lambda^{\frac{\nu-\mu-\Delta_0}{\Delta_0}}.$$

**Multipling by a Projector and Adding a definite sum:**

$$S(\nu) = \sum_{\mu_0=0}^{\Delta_0-1} \mathbf{Proj}(\nu, \mu_0, \Delta_0) \left[ \lambda^{\frac{\nu-\mu_0}{\Delta_0}} S(\mu_0) + \sum_{\mu=\mu_0}^{\nu-\Delta_0} \mathbf{Proj}(\mu, \mu_0, \Delta_0) \Delta S(\mu) \lambda^{\frac{\nu-\mu-\Delta_0}{\Delta_0}} \right].$$

**Rewriting the Inner Sum Range and Symplifying**

$$S(\nu) = \lambda^{\frac{\nu}{\Delta_0}} \sum_{\mu_0=0}^{\Delta_0-1} \mathrm{Proj}(\nu, \mu_0, \Delta_0) \left[ \lambda^{-\frac{\mu_0}{\Delta_0}} S(\mu_0) + \sum_{\substack{\mu=0 \\ \uparrow}}^{\nu-1} \mathrm{Proj}(\mu, \mu_0, \Delta_0) \Delta S(\mu) \lambda^{-\frac{\mu+\Delta_0}{\Delta_0}} \right].$$

# Step II——Key Step(Continued)

$$S(\nu) = \lambda^{\frac{\nu-\nu_0}{\Delta_0}} S(\nu_0) + \sum_{\mu=\nu_0}^{\nu-\Delta_0} \text{Proj}(\mu, \nu_0, \Delta_0) \Delta S(\mu) \lambda^{\frac{\nu-\mu-\Delta_0}{\Delta_0}}$$

**Replacing $\nu_0$ by $\mu_0$:**

$$S(\nu) = \lambda^{\frac{\nu-\mu_0}{\Delta_0}} S(\mu_0) + \sum_{\mu=\mu_0}^{\nu-\Delta_0} \textbf{Proj}(\mu, \mu_0, \Delta_0) \Delta S(\mu) \lambda^{\frac{\nu-\mu-\Delta_0}{\Delta_0}}.$$

**Multipling by a Projector and Adding a definite sum:**

$$S(\nu) = \sum_{\mu_0=0}^{\Delta_0-1} \textbf{Proj}(\nu, \mu_0, \Delta_0) \left[\lambda^{\frac{\nu-\mu_0}{\Delta_0}} S(\mu_0) + \sum_{\mu=\mu_0}^{\nu-\Delta_0} \textbf{Proj}(\mu, \mu_0, \Delta_0) \Delta S(\mu) \lambda^{\frac{\nu-\mu-\Delta_0}{\Delta_0}}\right].$$

**Rewriting the Inner Sum Range and Symplifying**

$$S(\nu) = \lambda^{\frac{\nu}{\Delta_0}} \sum_{\mu_0=0}^{\Delta_0-1} \text{Proj}(\nu, \mu_0, \Delta_0) \left[\lambda^{-\frac{\mu_0}{\Delta_0}} S(\mu_0) + \sum_{\substack{\mu=0 \\ \uparrow}}^{\nu-1} \text{Proj}(\mu, \mu_0, \Delta_0) \Delta S(\mu) \lambda^{-\frac{\mu+\Delta_0}{\Delta_0}}\right].$$

# Step II——Key Step(Continued)

$$S(\nu) = \lambda^{\frac{\nu-\nu_0}{\Delta_0}} S(\nu_0) + \sum_{\mu=\nu_0}^{\nu-\Delta_0} \text{Proj}(\mu, \nu_0, \Delta_0)\, \Delta S(\mu)\, \lambda^{\frac{\nu-\mu-\Delta_0}{\Delta_0}}$$

**Replacing $\nu_0$ by $\mu_0$:**

$$S(\nu) = \lambda^{\frac{\nu-\mu_0}{\Delta_0}} S(\mu_0) + \sum_{\mu=\mu_0}^{\nu-\Delta_0} \textbf{Proj}(\mu, \mu_0, \Delta_0)\, \Delta S(\mu)\, \lambda^{\frac{\nu-\mu-\Delta_0}{\Delta_0}}.$$

**Multipling by a Projector and Adding a definite sum:**

$$S(\nu) = \sum_{\mu_0=0}^{\Delta_0-1} \textbf{Proj}(\nu, \mu_0, \Delta_0) \left[ \lambda^{\frac{\nu-\mu_0}{\Delta_0}} S(\mu_0) + \sum_{\mu=\mu_0}^{\nu-\Delta_0} \textbf{Proj}(\mu, \mu_0, \Delta_0)\, \Delta S(\mu)\, \lambda^{\frac{\nu-\mu-\Delta_0}{\Delta_0}} \right].$$

**Rewriting the Inner Sum Range and Symplifying**

$$S(\nu) = \lambda^{\frac{\nu}{\Delta_0}} \sum_{\mu_0=0}^{\Delta_0-1} \text{Proj}(\nu, \mu_0, \Delta_0) \left[ \lambda^{-\frac{\mu_0}{\Delta_0}} S(\mu_0) + \sum_{\substack{\mu=0 \\ \uparrow}}^{\nu-1} \text{Proj}(\mu, \mu_0, \Delta_0)\, \Delta S(\mu)\, \lambda^{-\frac{\mu+\Delta_0}{\Delta_0}} \right].$$

# Step II——Key Step(Continued)

$$S(\nu) = \lambda^{\frac{\nu-\nu_0}{\Delta_0}} S(\nu_0) + \sum_{\mu=\nu_0}^{\nu-\Delta_0} \text{Proj}(\mu, \nu_0, \Delta_0)\, \Delta S(\mu)\, \lambda^{\frac{\nu-\mu-\Delta_0}{\Delta_0}}$$

**Replacing $\nu_0$ by $\mu_0$:**

$$S(\nu) = \lambda^{\frac{\nu-\mu_0}{\Delta_0}} S(\mu_0) + \sum_{\mu=\mu_0}^{\nu-\Delta_0} \textbf{Proj}(\mu, \mu_0, \Delta_0)\, \Delta S(\mu)\, \lambda^{\frac{\nu-\mu-\Delta_0}{\Delta_0}}.$$

**Multipling by a Projector and Adding a definite sum:**

$$S(\nu) = \sum_{\mu_0=0}^{\Delta_0-1} \textbf{Proj}(\nu, \mu_0, \Delta_0) \left[ \lambda^{\frac{\nu-\mu_0}{\Delta_0}} S(\mu_0) + \sum_{\mu=\mu_0}^{\nu-\Delta_0} \textbf{Proj}(\mu, \mu_0, \Delta_0)\, \Delta S(\mu)\, \lambda^{\frac{\nu-\mu-\Delta_0}{\Delta_0}} \right].$$

**Rewriting the Inner Sum Range and Symplifying**

$$S(\nu) = \lambda^{\frac{\nu}{\Delta_0}} \sum_{\mu_0=0}^{\Delta_0-1} \text{Proj}(\nu, \mu_0, \Delta_0) \left[ \lambda^{-\frac{\mu_0}{\Delta_0}} S(\mu_0) + \sum_{\substack{\mu=0 \\ \uparrow}}^{\nu-1} \text{Proj}(\mu, \mu_0, \Delta_0)\, \Delta S(\mu)\, \lambda^{-\frac{\mu+\Delta_0}{\Delta_0}} \right].$$

# Example

## Example

$$
\begin{aligned}
S(\nu) &= \sum_{x=1}^{\nu}\sum_{y=1}^{x}\frac{1}{(\nu-x+1)(x+y+1)} \\
&= \sum_{x=0}^{\nu-1}\frac{1}{3(1+x)}+\sum_{x=0}^{\nu-1}\sum_{y=0}^{x-1}\frac{1}{2(x+2)}\left[\frac{1}{y+1}-\frac{1}{y+2}\right] \\
&\quad -\sum_{x=0}^{\nu-1}\sum_{y=0}^{x-1}\frac{1}{x+3}\left[\frac{1}{y+1}+\frac{1}{y+3}\right] \\
&\quad +\sum_{x=0}^{\nu-1}\sum_{y=0}^{x-1}\frac{1}{2x+5}\left[\frac{1}{y+1}+\frac{1}{2y+5}\right]
\end{aligned}
$$

# Advantages of the Algorithm

Speed.

This is the reason for Carsten to have interest in this algorithm.

# Advantages of the Algorithm

Speed.
This is the reason for Carsten to have interest in this algorithm.

# Problem I —— Compatiblity with Sigma.m

## Problem I

Sigma.m can only deal with the cases when the upper bound of the inner sum is the index of the next outer sum.

$$\ldots f_{k+1}\left(i_{k+1}\right)\sum_{i_k=1}^{i_{k+1}} f_k\left(i_k\right)\ldots \checkmark$$

$$\ldots f_{k+1}\left(i_{k+1}\right)\sum_{i_k=1}^{i_{k+1}-1} f_k\left(i_k\right)\ldots \times$$

## Solution

Shift the sum index, ACCORDINGLY.

# Problem I —— Compatiblity with Sigma.m

## Problem I

Sigma.m can only deal with the cases when the upper bound of the inner sum is the index of the next outer sum.

$$\ldots f_{k+1}\left(i_{k+1}\right) \sum_{i_k=1}^{i_{k+1}} f_k\left(i_k\right) \ldots \checkmark$$

$$\ldots f_{k+1}\left(i_{k+1}\right) \sum_{i_k=1}^{i_{k+1}-1} f_k\left(i_k\right) \ldots \times$$

## Solution

Shift the sum index, ACCORDINGLY.

# Problem I —— Compatiblity with Sigma.m

## Problem I

Sigma.m can only deal with the cases when the upper bound of the inner sum is the index of the next outer sum.

$$\ldots f_{k+1}\left(i_{k+1}\right)\sum_{i_k=1}^{i_{k+1}} f_k\left(i_k\right)\ldots \checkmark$$

$$\ldots f_{k+1}\left(i_{k+1}\right)\sum_{i_k=1}^{i_{k+1}-1} f_k\left(i_k\right)\ldots \times$$

## Solution

Shift the sum index, ACCORDINGLY.

# Problem I —— Compatiblity with Sigma.m

## Problem I

Sigma.m can only deal with the cases when the upper bound of the inner sum is the index of the next outer sum.

$$\dots f_{k+1}\left(i_{k+1}\right)\sum_{i_k=1}^{i_{k+1}} f_k\left(i_k\right)\dots \checkmark$$

$$\dots f_{k+1}\left(i_{k+1}\right)\sum_{i_k=1}^{i_{k+1}-1} f_k\left(i_k\right)\dots \times$$

## Solution

Shift the sum index, ACCORDINGLY.

# Problem II——Root of Unity

Recall:

$$\mathrm{Proj}\,(m,n,d)=\frac{1}{d}\sum_{k=1}^{d}\exp\left(2\pi ik\frac{m-n}{d}\right)=\begin{cases}0 & m\not\equiv n\ (mod\ d)\\ 1 & m\equiv n\ (mod\ d)\end{cases}.$$

**Example**

$$\begin{cases}\mathrm{Proj}\,(m,n,1)\equiv 1\\ \mathrm{Proj}\,(m,n,2)=\frac{(-1)^m+(-1)^n}{2}\end{cases}$$

**When $d\geq 3$, root of unity is inevitable.**

**Example**

$$\underbrace{\mathrm{Proj}\,(6,2,3)=\frac{1}{3}\left(1+e^{-\frac{2\pi i}{3}}+e^{\frac{2\pi i}{3}}\right)}=0$$

Mathematica

# Problem II——Root of Unity

Recall:

$$\mathrm{Proj}\left(m,n,d\right)=\frac{1}{d}\sum_{k=1}^{d}\exp\left(2\pi ik\frac{m-n}{d}\right)=\begin{cases}0 & m\not\equiv n\ (mod\ d)\\ 1 & m\equiv n\ (mod\ d)\end{cases}.$$

**Example**

$$\begin{cases}\mathrm{Proj}\left(m,n,1\right)\equiv 1\\ \mathrm{Proj}\left(m,n,2\right)=\frac{(-1)^m+(-1)^n}{2}\end{cases}$$

**When $d\geq 3$, root of unity is inevitable.**

**Example**

$$\mathrm{Proj}\left(6,2,3\right)=\underbrace{\frac{1}{3}\left(1+e^{-\frac{2\pi i}{3}}+e^{\frac{2\pi i}{3}}\right)}_{\text{Mathematica}}=0$$

# Problem II——Root of Unity

Recall:

$$\text{Proj}\,(m,n,d) = \frac{1}{d}\sum_{k=1}^{d}\exp\left(2\pi i k\frac{m-n}{d}\right) = \begin{cases} 0 & m \not\equiv n\ (mod\ d) \\ 1 & m \equiv n\ (mod\ d) \end{cases}.$$

**Example**

$$\begin{cases} \text{Proj}\,(m,n,1) \equiv 1 \\ \text{Proj}\,(m,n,2) = \frac{(-1)^m+(-1)^n}{2} \end{cases}$$

**When $d \geq 3$, root of unity is inevitable.**

**Example**

$$\underbrace{\text{Proj}\,(6,2,3) = \frac{1}{3}\left(1+e^{-\frac{2\pi i}{3}}+e^{\frac{2\pi i}{3}}\right)}_{\text{Mathematica}} = 0$$

# Problem II——Root of Unity

Recall:

$$\text{Proj}\,(m,n,d) = \frac{1}{d}\sum_{k=1}^{d}\exp\left(2\pi ik\frac{m-n}{d}\right) = \begin{cases} 0 & m \not\equiv n\ (mod\ d) \\ 1 & m \equiv n\ (mod\ d) \end{cases}.$$

## Example

$$\begin{cases} \text{Proj}\,(m,n,1) \equiv 1 \\ \text{Proj}\,(m,n,2) = \frac{(-1)^m+(-1)^n}{2} \end{cases}$$

**When $d \geq 3$, root of unity is inevitable.**

## Example

$$\text{Proj}\,(6,2,3) = \underbrace{\frac{1}{3}\left(1+e^{-\frac{2\pi i}{3}}+e^{\frac{2\pi i}{3}}\right)}_{\text{Mathematica}} = 0$$

# Problem II——Root of Unity (Continued)

**FACT**

There is no other substitutions for the projector function.

**Solutions/Compromise**

1. Try to make $\Delta_0$ as small as possible. ($\Delta_0 = 1$ or $\Delta_0 = 2$)
2. "Replacement $\nu \mapsto \Delta_0 \nu$". (Conditionally)

We can only make such replacement for the outermost sum.

# Problem II——Root of Unity (Continued)

**FACT**

There is no other substitutions for the projector function.

**Solutions/Compromise**

1. Try to make $\Delta_0$ as small as possible. ($\Delta_0 = 1$ or $\Delta_0 = 2$)
2. "Replacement $\nu \mapsto \Delta_0 \nu$". (Conditionally)

We can only make such replacement for the outermost sum.

# Problem II——Root of Unity (Continued)

**FACT**

There is no other substitutions for the projector function.

**Solutions/Compromise**

1. Try to make $\Delta_0$ as small as possible. ($\Delta_0 = 1$ or $\Delta_0 = 2$)
2. "Replacement $\nu \mapsto \Delta_0 \nu$". (Conditionally)

We can only make such replacement for the outermost sum.

# Problem II——Root of Unity (Continued)

**FACT**

There is no other substitutions for the projector function.

**Solutions/Compromise**

1. Try to make $\Delta_0$ as small as possible. ($\Delta_0 = 1$ or $\Delta_0 = 2$)
2. "Replacement $\nu \mapsto \Delta_0\nu$". (Conditionally)

We can only make such replacement for the outermost sum.

# Problem II——Root of Unity (Continued)

**FACT**

There is no other substitutions for the projector function.

**Solutions/Compromise**

1. Try to make $\Delta_0$ as small as possible. ($\Delta_0 = 1$ or $\Delta_0 = 2$)
2. "Replacement $\nu \mapsto \Delta_0 \nu$". (Conditionally)

We can only make such replacement for the outermost sum.

# Problem II——Root of Unity (Continued)

## FACT

There is no other substitutions for the projector function.

## Solutions/Compromise

1. Try to make $\Delta_0$ as small as possible. ($\Delta_0 = 1$ or $\Delta_0 = 2$)
2. "Replacement $\nu \mapsto \Delta_0 \nu$". (Conditionally)

We can only make such replacement for the outermost sum.

# Problem III——Poles

**FACT**

It is very common when converting sums into nested sums.

**Example**

[Created by PFD]

$$S(\nu) = \sum_{x=1}^{\nu} \frac{1}{(\nu + x)(x+1)} \Rightarrow S(1) = \frac{1}{4}$$

$$\begin{aligned} \bar{S}(\nu) &= \sum_{x=1}^{\nu} \left[ \frac{1}{(\nu - 1)(\nu + x)} - \frac{1}{(\nu - 1)(x+1)} \right] \\ &= \frac{1}{\nu - 1} \sum_{x=0}^{\nu-1} \left[ \frac{1}{x+2} + \frac{1}{x+1} - \frac{1}{2x+1} - \frac{1}{2x+2} \right] \\ \bar{S}(1) &= \frac{0}{0} = \text{"Indetermined"} \end{aligned}$$

# Problem III——Poles

## FACT

It is very common when converting sums into nested sums.

## Example

[Created by PFD]

$$S(\nu) = \sum_{x=1}^{\nu} \frac{1}{(\nu + x)(x+1)} \Rightarrow S(1) = \frac{1}{4}$$

$$\begin{aligned} \bar{S}(\nu) &= \sum_{x=1}^{\nu} \left[ \frac{1}{(\nu-1)(\nu+x)} - \frac{1}{(\nu-1)(x+1)} \right] \\ &= \frac{1}{\nu-1} \sum_{x=0}^{\nu-1} \left[ \frac{1}{x+2} + \frac{1}{x+1} - \frac{1}{2x+1} - \frac{1}{2x+2} \right] \\ \bar{S}(1) &= \frac{0}{0} = \text{"Indetermined"} \end{aligned}$$

# Problem III——Poles

## FACT

It is very common when converting sums into nested sums.

## Example

[Created by PFD]

$$S(\nu) = \sum_{x=1}^{\nu} \frac{1}{(\nu+x)(x+1)} \Rightarrow S(1) = \frac{1}{4}$$

$$\begin{aligned} \bar{S}(\nu) &= \sum_{x=1}^{\nu} \left[ \frac{1}{(\nu-1)(\nu+x)} - \frac{1}{(\nu-1)(x+1)} \right] \\ &= \frac{1}{\nu-1} \sum_{x=0}^{\nu-1} \left[ \frac{1}{x+2} + \frac{1}{x+1} - \frac{1}{2x+1} - \frac{1}{2x+2} \right] \\ \bar{S}(1) &= \frac{0}{0} = \text{"Indetermined"} \end{aligned}$$

# Problem III——Poles (Continued)

## Example

[Created by Shifting (Crossing the Boundary)]

$$S(\nu) = \sum_{x=1}^{\nu} \sum_{z=1}^{x-1} \frac{1}{(\nu + x + 1)(x + z)}.$$

**Note:**

$$\Delta_0 = 1, \Delta_x = \Delta_1 = -1, \Delta_z = \Delta_2 = 1.$$

## Fact

*While computing $\Delta S(\nu)$, the following sum appears*

$$\sum_{x=1}^{\nu} \sum_{z=(x-1)+1}^{(x-1)+(-1)-1} \frac{1}{(\nu + x + 1)(x + z)} = -\sum_{x=1}^{\nu} \left[ \frac{1}{(\nu + x + 1)(2x - 1)} + \frac{1}{(\nu + x + 1)(2x - 2)} \right],$$

# Problem III——Poles (Continued)

## Example

[Created by Shifting (Crossing the Boundary)]

$$S(\nu) = \sum_{x=1}^{\nu} \sum_{z=1}^{x-1} \frac{1}{(\nu + x + 1)(x + z)}.$$

**Note:**

$$\Delta_0 = 1, \Delta_x = \Delta_1 = -1, \Delta_z = \Delta_2 = 1.$$

## Fact

*While computing $\Delta S(\nu)$, the following sum appears*

$$\sum_{x=1}^{\nu} \sum_{z=(x-1)+1}^{(x-1)+(-1)-1} \frac{1}{(\nu + x + 1)(x + z)} = -\sum_{x=1}^{\nu} \left[ \frac{1}{(\nu + x + 1)(2x - 1)} + \frac{1}{(\nu + x + 1)(2x - 2)} \right],$$

# Dealing with Poles—by Authors

## Remark

In the paper, during Step II, when poles/divergence terms are created, they introduce regularization parameter, by $i_k \mapsto i_k + \delta_k$. Also, the paper talks about Step III, evaluation, which one can eventually send $\delta_k \to 0$.

## Example

$$\sum_{x \geq y \geq z \geq 1} \frac{1}{2x(x-y)(x+z)} \mapsto \sum_{x \geq y \geq z \geq 1} \frac{1}{2x(x-y+\delta)(x+z)}.$$

## FACT

The speed will be slowed down dramatically by introducing more parameters, which violates our original purpose for implementation.

# Dealing with Poles—by Authors

## Remark

In the paper, during Step II, when poles/divergence terms are created, they introduce regularization parameter, by $i_k \mapsto i_k + \delta_k$. Also, the paper talks about Step III, evaluation, which one can eventually send $\delta_k \to 0$.

## Example

$$\sum_{x \geq y \geq z \geq 1} \frac{1}{2x\left(x-y\right)\left(x+z\right)} \mapsto \sum_{x \geq y \geq z \geq 1} \frac{1}{2x\left(x-y+\delta\right)\left(x+z\right)}.$$

## FACT

The speed will be slowed down dramatically by introducing more parameters, which violates our original purpose for implementation.

# Dealing with Poles—by Authors

## Remark

In the paper, during Step II, when poles/divergence terms are created, they introduce regularization parameter, by $i_k \mapsto i_k + \delta_k$. Also, the paper talks about Step III, evaluation, which one can eventually send $\delta_k \to 0$.

## Example

$$\sum_{x \geq y \geq z \geq 1} \frac{1}{2x(x-y)(x+z)} \mapsto \sum_{x \geq y \geq z \geq 1} \frac{1}{2x(x-y+\delta)(x+z)}.$$

## FACT

The speed will be slowed down dramatically by introducing more parameters, which violates our original purpose for implementation.

# Dealing with Poles—Modifying the Starting Point

**[Q]:**Recall the expression

$$S(\nu) = \lambda^{\frac{\nu}{\Delta_0}} \sum_{\mu_0=0}^{\Delta_0-1} \mathrm{Proj}(\nu, \mu_0, \Delta_0) \left[ \lambda^{-\frac{\mu_0}{\Delta_0}} S(\mu_0) + \sum_{\mu=0}^{\nu-1} \mathrm{Proj}(\mu, \mu_0, \Delta_0) \Delta S(\mu) \lambda^{-\frac{\mu+\Delta_0}{\Delta_0}} \right],$$

and suppose $\Delta S(\mu)$ has a pole at $\mu = 1$. How to deal with it?
RECALL

$$\begin{aligned} f(101) &= f(2) + \sum_{k=0}^{32} \Delta f(3k+2) \\ &= f(5) + \sum_{k=1}^{32} \Delta f(3k+2) \\ &= \ldots \end{aligned}$$

# Dealing with Poles—Modifying the Starting Point

**[Q]:**Recall the expression

$$S(\nu) = \lambda^{\frac{\nu}{\Delta_0}} \sum_{\mu_0=0}^{\Delta_0-1} \mathrm{Proj}(\nu, \mu_0, \Delta_0) \left[ \lambda^{-\frac{\mu_0}{\Delta_0}} S(\mu_0) + \sum_{\mu=0}^{\nu-1} \mathrm{Proj}(\mu, \mu_0, \Delta_0) \Delta S(\mu) \lambda^{-\frac{\mu+\Delta_0}{\Delta_0}} \right],$$

and suppose $\Delta S(\mu)$ has a pole at $\mu = 1$. How to deal with it?

RECALL

$$\begin{aligned} f(101) &= f(2) + \sum_{k=0}^{32} \Delta f(3k+2) \\ &= f(5) + \sum_{k=1}^{32} \Delta f(3k+2) \\ &= \ldots \end{aligned}$$

# Dealing with Poles—Modifying the Starting Point

**[Q]:**Recall the expression

$$S(\nu) = \lambda^{\frac{\nu}{\Delta_0}} \sum_{\mu_0=0}^{\Delta_0-1} \mathrm{Proj}(\nu, \mu_0, \Delta_0) \left[ \lambda^{-\frac{\mu_0}{\Delta_0}} S(\mu_0) + \sum_{\mu=0}^{\nu-1} \mathrm{Proj}(\mu, \mu_0, \Delta_0) \Delta S(\mu) \lambda^{-\frac{\mu+\Delta_0}{\Delta_0}} \right],$$

and suppose $\Delta S(\mu)$ has a pole at $\mu = 1$. How to deal with it?
RECALL

$$\begin{aligned} f(101) &= f(2) + \sum_{k=0}^{32} \Delta f(3k+2) \\ &= f(5) + \sum_{k=1}^{32} \Delta f(3k+2) \\ &= \ldots \end{aligned}$$

# Dealing with Poles—Modifying the Starting Point

**[Q]:**Recall the expression

$$S(\nu) = \lambda^{\frac{\nu}{\Delta_0}} \sum_{\mu_0=0}^{\Delta_0-1} \mathrm{Proj}(\nu, \mu_0, \Delta_0) \left[ \lambda^{-\frac{\mu_0}{\Delta_0}} S(\mu_0) + \sum_{\mu=0}^{\nu-1} \mathrm{Proj}(\mu, \mu_0, \Delta_0) \Delta S(\mu) \lambda^{-\frac{\mu+\Delta_0}{\Delta_0}} \right],$$

and suppose $\Delta S(\mu)$ has a pole at $\mu = 1$. How to deal with it?

RECALL

$$\begin{aligned} f(101) &= f(2) + \sum_{k=0}^{32} \Delta f(3k+2) \\ &= f(5) + \sum_{k=1}^{32} \Delta f(3k+2) \\ &= \ldots \end{aligned}$$

# Dealing with Poles—Modifying the Starting Point (Continued)

## KEY

Find $k \in \mathbb{N}$, s.t. $k \cdot \Delta_0 > 1$(the largest pole of $\Delta S(\mu)$ & $S(\nu_0)$).

## Corresponding Expression

$$S(\nu) = \lambda^{\frac{\nu}{\Delta_0}} \sum_{\mu_0=0}^{\Delta_0-1} \mathrm{Proj}(\nu, \mu_0, \Delta_0) \left[ \lambda^{-\frac{\mu_0}{\Delta_0}-k} S(\mu_0 + k \cdot \Delta_0) + \sum_{\mu=k\Delta_0}^{\nu-1} \mathrm{Proj}(\mu, \mu_0, \Delta_0) \Delta S(\mu) \lambda^{-\frac{\mu+\Delta_0}{\Delta_0}} \right]$$

# Dealing with Poles—Modifying the Starting Point (Continued)

**KEY**

Find $k \in \mathbb{N}$, s.t. $k \cdot \Delta_0 > 1$(the largest pole of $\Delta S(\mu)$ & $S(\nu_0)$).

**Corresponding Expression**

$$S(\nu) = \lambda^{\frac{\nu}{\Delta_0}} \sum_{\mu_0=0}^{\Delta_0-1} \text{Proj}(\nu, \mu_0, \Delta_0) \left[ \lambda^{-\frac{\mu_0}{\Delta_0}-k} S(\mu_0 + k \cdot \Delta_0) + \sum_{\mu=k\Delta_0}^{\nu-1} \text{Proj}(\mu, \mu_0, \Delta_0) \Delta S(\mu) \lambda^{-\frac{\mu+\Delta_0}{\Delta_0}} \right]$$

# Dealing with Poles—Modifying the Starting Point (Continued)

## KEY

Find $k \in \mathbb{N}$, s.t. $k \cdot \Delta_0 > 1$(the largest pole of $\Delta S(\mu)$ & $S(\nu_0)$).

## Corresponding Expression

$$S(\nu) = \lambda^{\frac{\nu}{\Delta_0}} \sum_{\mu_0=0}^{\Delta_0-1} \text{Proj}(\nu, \mu_0, \Delta_0) \left[ \lambda^{-\frac{\mu_0}{\Delta_0}-k} S(\mu_0 + k \cdot \Delta_0) + \sum_{\mu=k\Delta_0}^{\nu-1} \text{Proj}(\mu, \mu_0, \Delta_0) \Delta S(\mu) \lambda^{-\frac{\mu+\Delta_0}{\Delta_0}} \right]$$

# Dealing with Poles—Splitting the Sums

## Example

$$S(\nu) := \sum_{x=1}^{\nu}\sum_{y=1}^{x} \frac{1}{(x+y)(y+1)}$$
$$= \sum_{x=1}^{\nu}\sum_{y=1}^{x} \left[\frac{1}{(x-1)(x+y)} - \frac{1}{(x-1)(y+1)}\right]$$

$$\begin{cases} S_1(\nu) = \sum_{x=1}^{\nu}\sum_{y=1}^{x} \frac{1}{(x+y)(y+1)}\Big|_{x=1} = \frac{1}{4} \\ S_2(\nu) = \sum_{x=2}^{\nu}\sum_{y=1}^{x} \frac{1}{(x+y)(y+1)} \end{cases}$$

# Dealing with Poles—Splitting the Sums

## Example

$$S(\nu) := \sum_{x=1}^{\nu}\sum_{y=1}^{x}\frac{1}{(x+y)(y+1)}$$

$$= \sum_{x=1}^{\nu}\sum_{y=1}^{x}\left[\frac{1}{(x-1)(x+y)} - \frac{1}{(x-1)(y+1)}\right]$$

$$\begin{cases} S_1(\nu) = \sum_{x=1}^{\nu}\sum_{y=1}^{x}\frac{1}{(x+y)(y+1)}\Big|_{x=1} = \frac{1}{4} \\ S_2(\nu) = \sum_{x=2}^{\nu}\sum_{y=1}^{x}\frac{1}{(x+y)(y+1)} \end{cases}$$

# Dealing with Poles—Splitting the Sums

## Example

$$S(\nu) := \sum_{x=1}^{\nu}\sum_{y=1}^{x}\frac{1}{(x+y)(y+1)}$$

$$= \sum_{x=1}^{\nu}\sum_{y=1}^{x}\left[\frac{1}{(x-1)(x+y)}-\frac{1}{(x-1)(y+1)}\right]$$

$$\begin{cases} S_1(\nu) = \sum_{x=1}^{\nu}\sum_{y=1}^{x}\frac{1}{(x+y)(y+1)}\Big|_{x=1} = \frac{1}{4} \\ S_2(\nu) = \sum_{x=2}^{\nu}\sum_{y=1}^{x}\frac{1}{(x+y)(y+1)} \end{cases}$$

# Conclusion

## Conclusions/Experience

1. This algorithm works faster than the usual/general method of Sigma.m.
2. Root of unity is inherited from the algorithm. It seems to be impossible to avoid.
3. Poles are very common while converting sums (creating telescoping). And the poles created by PFD is easier to deal with comparing to those from shifting.
4. Adding regularization parameters will slow down this algorithm dramatically.

# Conclusion

## Conclusions/Experience

1. This algorithm works faster than the usual/general method of Sigma.m.
2. Root of unity is inherited from the algorithm. It seems to be impossible to avoid.
3. Poles are very common while converting sums (creating telescoping). And the poles created by PFD is easier to deal with comparing to those from shifting.
4. Adding regularization parameters will slow down this algorithm dramatically.

# Conclusion

## Conclusions/Experience

1. This algorithm works faster than the usual/general method of Sigma.m.
2. Root of unity is inherited from the algorithm. It seems to be impossible to avoid.
3. Poles are very common while converting sums (creating telescoping). And the poles created by PFD is easier to deal with comparing to those from shifting.
4. Adding regularization parameters will slow down this algorithm dramatically.

# Conclusion

## Conclusions/Experience

1. This algorithm works faster than the usual/general method of Sigma.m.
2. Root of unity is inherited from the algorithm. It seems to be impossible to avoid.
3. Poles are very common while converting sums (creating telescoping). And the poles created by PFD is easier to deal with comparing to those from shifting.
4. Adding regularization parameters will slow down this algorithm dramatically.

# THE END

Thank you!