Binomial Identity in Arbitrary Bases

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Introduction

Binomial Identity

$$(X+Y)^n = \sum_{k=0}^{n} \binom{n}{k} X^k Y^{n-k}.$$

Generalization.

Multi-nomial Identity

$$(X_1+\cdots+X_m)^n = \sum_{k_1+\cdots+k_m=n} \binom{n}{k_1,\ldots,k_m} X_1^{k_1}\cdots X_m^{k_m}.$$

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Binary

n	0	1	2	3	4	5	6	7	...
$(n)_2$	0	1	10	11	100	101	110	111	...

DEF.

$$S_2(n) = \#\text{ of 1's in the binary expansion of } n.$$

Example

$$\begin{cases} S_2(3) = 2 \\ S_2(4) = 1 \\ S_2(7) = 3 \end{cases}$$

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Theorem

$$(X+Y)^{S_2(n)} = \sum_{0 \le k \le n \&\& \text{some condition}} X^{S_2(k)} Y^{S_2(n-k)},$$

some condition = there is no carry when adding $k + (n-k) = n$.

In fact,

$$k + (n-k) = n \text{ is carry free} \Leftrightarrow S_2(k) + S_2(n-k) = S_2(n).$$

Thus,

$$(X+Y)^{S_2(n)} = \sum_{S_2(k)+S_2(n-k)=S_2(n)} X^{S_2(k)} Y^{S_2(n-k)}.$$

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$n = 6 = (110)_2$. Therefore,

$$LHS = (X+Y)^{S_2(2)} = (X+Y)^2.$$

On the other hand,

$k+(n-k)$	$0+6$	$1+5$	$2+4$	$3+3$	$4+2$	$5+1$	$6+0$
Binary	000 110 110	001 101 110	010 100 110	011 011 110	100 010 110	101 001 110	110 000 110
Carry-free	✓	x	✓	x	✓	x	✓

$$\begin{aligned} RHS &= X^{S_2(0)}Y^{S_2(6)} + X^{S_2(2)}Y^{S_2(4)} + X^{S_2(4)}Y^{S_2(2)} + X^{S_2(6)}Y^{S_2(0)} \\ &= Y^2 + XY + XY + X^2. \end{aligned}$$

Remark

Carry Free $\Leftrightarrow$ $1+1$ does not appear.

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$$(X+Y)^{S_2(n)} = \sum_{S_2(k)+S_2(n-k)=S_2(n)} X^{S_2(k)} Y^{S_2(n-k)}.$$

carry-free$\Leftrightarrow$ $\binom{S_2(n)}{l}$, i.e., distributing 1's in $(n)_2$.

- How to generalize this result to other bases rather than 2, for example 3?
- How to define

$$S_3(n) = ???$$

  \# of 1's? # of 2's? # of non-zero digits?
- In binary, $0+1$, $0+0$ are OK and $1+1$ is not. $\Leftrightarrow$Carry Free. In base 3,

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Solutions

DEF.

$S_2(n) = \#$ of 1's in the binary expansion of $n =$ sum of all digits.

Thus,

$$S_b(n) = \text{sum of all digits of } n \text{ in its expansion of base } b.$$

This also implies $1+1=2$ is allowed.

Unfortunately $(X+Y)^{S_3(n)} \neq \sum_{\text{carry-free}} X^{S_3(k)} Y^{S_3(n-k)}$

$S_3(2) = S_3((2)_3) = 2.$

$k+(n-k)$	$0+2$	$1+1$	$2+0$
Binary	$\begin{array}{r}0\\ \underline{2}\\ 2\end{array}$	$\begin{array}{r}1\\ \underline{1}\\ 2\end{array}$	$\begin{array}{r}2\\ \underline{0}\\ 2\end{array}$
Carry-free	✓	✓	✓

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Carry-free	✓	✓	✓

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Result

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$$(X+Y)^{S_b(n)}=\sum_{\square_1}\square_2 X^{S_b(k)}Y^{S_b(n-k)}$$

$$\begin{cases}\square_1=\text{carry-free} & \mapsto ?\\ \square_2=1 & \mapsto ?\end{cases}$$

Modify $\square_2$:

$$(X+Y)^{S_b(n)}=\sum_{\text{carry-free}}\left(\prod_{j=0}^{b-1}(j!)^{S_b^{(j)}(n)-S_b^{(j)}(k)-S_b^{(j)}(n-k)}\right)X^{S_b(k)}Y^{S_b(n-k)},$$

where

$$S_b^{(j)}(m)=\#\text{ of } j\text{'s in } m.$$

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$$(X+Y)^{S_b(n)} = \sum_{\text{carry-free}} \left(\prod_{j=0}^{b-1} (j!)^{S_b^{(j)}(n) - S_b^{(j)}(k) - S_b^{(j)}(n-k)} \right) X^{S_b(k)} Y^{S_b(n-k)},$$

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Result

$n=2$

$S_3(2)=2.$

$k+(n-k)$	$0+2$	$1+1$	$2+0$
Binary	0 / $\frac{2}{2}$	1 / $\frac{1}{2}$	2 / $\frac{0}{2}$
Carry-free	✓	✓	✓

$$\begin{cases} LHS = (X+Y)^2 \\ RHS = X^2 + Y^2 + (2!)^{1-0-0} XY \end{cases}$$

Next

Define the $b-$ary binomial coefficients:

$$\binom{n}{k}_b := \begin{cases} \prod_{j=0}^{b-1} (j!)^{S_b^{(j)}(n) - S_b^{(j)}(k) - S_b^{(j)}(n-k)} & \text{if carry-free} \\ 0 & \text{otherwise} \end{cases}$$

$$(X+Y)^{S_b(n)} = \sum_{k=0}^{n} \binom{n}{k}_b X^{S_b(k)} Y^{S_b(n-k)}.$$

Result

$n = 2$

$S_3(2) = 2.$

$k + (n-k)$	$0+2$	$1+1$	$2+0$
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$$\binom{n}{k}_b := \begin{cases} \prod_{j=0}^{b-1} (j!)^{S_b^{(j)}(n) - S_b^{(j)}(k) - S_b^{(j)}(n-k)} & \text{if carry-free} \\ 0 & \text{otherwise} \end{cases}$$

$$(X+Y)^{S_b(n)} = \sum_{k=0}^{n} \binom{n}{k}_b X^{S_b(k)} Y^{S_b(n-k)}.$$

Result

$n = 2$

$S_3(2) = 2.$

$k + (n-k)$	$0+2$	$1+1$	$2+0$
Binary	$\begin{array}{c}0\\2\\\hline 2\end{array}$	$\begin{array}{c}1\\1\\\hline 2\end{array}$	$\begin{array}{c}2\\0\\\hline 2\end{array}$
Carry-free	✓	✓	✓

$$\begin{cases} LHS = (X+Y)^2 \\ RHS = X^2 + Y^2 + (2!)^{1-0-0} XY \end{cases}$$

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Triangle

$b = 4$

```
1
1 1
1 2 1
1 3 3 1
1 . . . 1
1 1 . . 1 1
1 2 1 . 1 2 1
1 3 3 1 1 3 3 1
1 . . . 2 . . . 1
1 1 . . 2 2 . . 1 1
1 2 1 . 2 4 2 . 1 2
1 3 3 1 2 6 6 2 1 3 3
1 . . . 3 . . . 3 . .
1 1 . . 3 3 . . 3 3 . .
1 2 1 . 3 6 3 . 3 6 3 .
1 3 3 1 3 9 9 3 3 9 9 3 1
```

b-ary Binomial Coefficients

Triangle

$b = 4$

1

1 1

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1 3 3 1 1 3 3 1

1 . . . 2 . . . 1

1 1 . . 2 2 . . 1 1

1 2 1 . 2 4 2 . 1 2

1 3 3 1 2 6 6 2 1 3 3

1 . . . 3 . . . 3 . .

1 1 . . 3 3 . . 3 3 . .

1 2 1 . 3 6 3 . 3 6 3 .

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1 . . . 2 . . . 1

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1 2 1 . 2 4 2 . 1 2

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b-ary Binomial Coefficients

Generating Function

Define

$$f(n,b,x) := \sum_{k=0}^{n} \binom{n}{k}_b x^k,$$

for $b = 4$,

n	$f(n,4,x)$
1	$1+x$
2	$(1+x)^2$
3	$(1+x)^3$
4	$1+x^4$
5	$(1+x)\left(1+x^4\right)$
6	$(1+x)^2\left(1+x^4\right)$
7	$(1+x)^3\left(1+x^4\right)$
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$$(X+Y)^{S_b(n)} = \sum_{k=0}^{n} \binom{n}{k}_b X^{S_b(k)} Y^{S_b(n-k)},$$

where

$$\binom{n}{k}_b = \left\{ \begin{matrix} \prod_{j=0}^{b-1} (j!)^{S_b^{(j)}(n) - S_b^{(j)}(k) - S_b^{(j)}(n-k)} & \text{carry-free} \\ 0 & \text{otherwise} \end{matrix} \right\} = \prod_{l=0}^{N-1} \binom{n_l}{k_l}.$$

In base b,

$$n = n_{N-1} n_{N-2} \cdots n_0 \text{ and } k = k_{N-1} k_{N-2} \cdots k_0$$

i.e.,

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Implies:

$$\sum_{k=0}^{n}\binom{n}{k}_b x^k=\prod_{l=0}^{N}\left(1+x^{b^l}\right)^{n_l}.$$

Remark[Lucas Theorem]

For a prime p,

$$\binom{n}{k}_p\equiv\prod_{l=0}^{N-1}\binom{n_l}{k_l}\pmod{p}.$$

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Thank You!