The Method of Brackets (MoB)

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June 22, 2016

Acknowledgement

Joint Work with:

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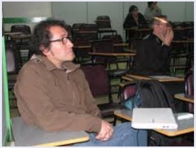
Karen Kohl



Ivan Gonzalez

Outlines

Rules

Idea

MoB evaluates $\int_0^\infty f(x)\,dx$ (most of the time) in terms of SERIES, with *ONLY SIX* rules:

Defintion [Indicator]

$$\phi_n := \frac{(-1)^n}{n!} = \frac{(-1)^n}{\Gamma(n+1)}$$

and

$$\phi_{1,\ldots,r} := \phi_{n_1,\ldots,n_r} = \phi_{n_1}\phi_{n_2}\cdots\phi_{n_r} = \prod_{i=1}^{r}\phi_{n_i}.$$

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P_1: $f(x) = \sum_{n=0}^{\infty} a_n x^{\alpha n+\beta-1} \Rightarrow \int_0^\infty f(x)\,dx \mapsto \sum_n a_n \langle \alpha n + \beta \rangle$—Bracket Series;

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P_3: For each bracket series, we assign <u>*index=# of sums− # of brackets*</u>;

E_1: $\sum_n \phi_n f(n) \langle \alpha n + \beta \rangle = \frac{1}{|\alpha|} f(n^*) \Gamma(-n^*)$,where n^* solves $\alpha n + \beta = 0$;

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$$\sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} f(n_1,\ldots,n_r) \prod_{i=1}^{r} \langle a_{i1}n_1 + \cdots + a_{ir}n_r + c_i \rangle = \frac{f(n_1^*,\ldots,n_r^*) \prod_{i=1}^{r} \Gamma(-n_i^*)}{|\det A|},$$

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E_3: The value of a multi-dimensional bracket series of ***POSITIVE*** index is obtained by computing all the contributions of maximal rank by Rule E_2. These contributions to the integral appear as series in the free parameters. Series converging in a common region are added and divergent series are discarded. Any series producing a non-real contribution is also discarded.

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Examples

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- P_1: $\frac{1}{1+x^2}=\sum_n(-1)^n x^{2n}\Rightarrow I=\sum_n(-1)^n\langle 2n+1\rangle=\sum_n\phi_n\Gamma(n+1)\langle 2n+1\rangle$
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Examples

$$I := \int_0^\infty x J_0(xy) \frac{dx}{\sqrt{a^2+x^2}} = y^{-1} e^{-ay} \quad [y>0 \; Re(a)>0]$$

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$$I = -\frac{\sinh(ay)}{y}.$$

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Rules (P-Production; E-Evaluation) $I = \int_0^\infty f(x)\,dx$

P_1: $f(x) = \sum_{n=0}^{\infty} a_n x^{\alpha n+\beta-1} \Rightarrow \int_0^\infty f(x)\,dx \mapsto \sum_n a_n \langle \alpha n + \beta \rangle$—Bracket Series;

P_2: $(a_1 + \cdots + a_r)^\alpha \mapsto \sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} a_1^{n_1} \cdots a_r^{n_r} \frac{\langle -\alpha + n_1 + \cdots + n_r \rangle}{\Gamma(-\alpha)}$;

P_3: For each bracket series, we assign index=# of sums− # of brackets;

E_1: $\sum_n \phi_n f(n) \langle \alpha n + \beta \rangle = \frac{1}{|\alpha|} f(n^*)\,\Gamma(-n^*)$, where n^* solves $\alpha n + \beta = 0$;

E_2: $$\sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} f(n_1,\ldots,n_r) \prod_{i=1}^{r} \langle a_{i1} n_1 + \cdots + a_{ir} n_r + c_i \rangle = \frac{f(n_1^*,\ldots,n_r^*) \prod_{i=1}^{r} \Gamma(-n_i^*)}{|\det A|},$$

$$(n_1^*,\ldots,n_r^*) \text{ solves } \begin{cases} a_{11} n_1 + \cdots + a_{1r} n_r + c_1 & = 0 \\ \ldots & \cdots \\ a_{r1} n_1 + \cdots + a_{rr} n_r + c_r & = 0 \end{cases};$$

E_3: The value of a multi-dimensional bracket series of ***POSITIVE*** index is obtained by computing all the contributions of maximal rank by Rule E_2. These contributions to the integral appear as series in the free parameters. Series converging in a common region are added and divergent series are discarded. Any series producing a non-real contribution is also discarded.

Ramanujan's Master Theorem[RMT]

Theorem[RMT]

$$\int_0^{\infty} x^{s-1}\left\{f(0)-\frac{f(1)}{1!}x+\frac{f(2)}{2!}x^2-\cdots\right\}dx=f(-s)\Gamma(s)$$

Remark

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$$\int_0^{\infty} x^{s-1}\left(\sum_{n=0}^{\infty}\phi_n f(n)x^n\right)dx=\sum_n \phi_n f(n)\langle n+s\rangle=f(-s)\Gamma(s)$$

(2) [Hardy]

•$H(\delta):=\{s=\sigma+\iota t:\sigma\geq-\delta,0<\delta<1\}$;

•$\psi(x)\in C^{\infty}(H(\delta))$; $\exists C,P,A,\ A<\pi$ such that $|\psi(s)|\leq Ce^{P\delta+A|t|}$, $\forall s\in H(\delta)$;

•$0<c<\delta,\ \Psi(x):=\frac{1}{2\pi\iota}\int_{c-\iota\infty}^{c+\iota\infty}\frac{\pi}{\sin(\pi s)}\psi(-s)x^{-s}dx \overset{0\leq x\leq e^{-P}}{=\!=\!=\!=} \sum_{k=0}^{\infty}\psi(k)(-x)^k$;

$$\int_0^{\infty}\Psi(x)x^{s-1}dx=\frac{\pi}{\sin(\pi s)}\psi(-s).$$

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Ramanujan's Master Theorem[RMT]

Theorem[RMT]

$$\int_0^\infty x^{s-1}\left\{f(0)-\frac{f(1)}{1!}x+\frac{f(2)}{2!}x^2-\cdots\right\}dx=f(-s)\Gamma(s)$$

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Rules Again

P_1: $f(x) = \sum_{n=0}^{\infty} a_n x^{\alpha n+\beta-1} \Rightarrow \int_0^\infty f(x)\,dx \mapsto \sum_n a_n \langle \alpha n + \beta \rangle$ $\boxed{s-1 \mapsto s}$

P_2: $(a_1 + \cdots + a_r)^\alpha \mapsto \sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} a_1^{n_1} \cdots a_r^{n_r} \frac{\langle -\alpha + n_1 + \cdots + n_r \rangle}{\Gamma(-\alpha)}$; $\boxed{\text{Next page}}$

P_3: Index=# of sums− # of brackets; $\boxed{\text{Just a definition}}$

E_1: $\sum_n \phi_n f(n) \langle \alpha n + \beta \rangle = \frac{1}{|\alpha|} f(n^*) \Gamma(-n^*)$ $\boxed{\text{RMT+Change of Variable}}$

E_2: $\boxed{\text{Iteration of RMT}}$

$$\sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} f(n_1,\ldots,n_r) \prod_{i=1}^{r} \langle a_{i1} n_1 + \cdots + a_{ir} n_r + c_i \rangle = \frac{f(n_1^*,\ldots,n_r^*) \prod_{i=1}^{r} \Gamma(-n_i^*)}{|\det A|}$$

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Rule P_2

$$
\begin{aligned}
& \frac{\Gamma(-\alpha)}{(a_1+\cdots+a_r)^{-\alpha}}. \\
= & \int_0^\infty x^{-\alpha-1} e^{-(a_1+\cdots+a_r)x} dx \\
= & \int_0^\infty x^{-\alpha-1} e^{-a_1 x} e^{-a_2 x} \cdots e^{-a_r x} dx \\
= & \int_0^\infty x^{-\alpha-1} \prod_{i=1}^{r} \left(\sum_{n_i=0}^{\infty} \phi_{n_i} (ax)^{n_i} \right) dx \\
= & \int_0^\infty \sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} a_1^{n_1} \cdots a_r^{n_r} x^{n_1+\cdots+n_r-\alpha-1} dx \\
= & \sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} a_1^{n_1} \cdots a_r^{n_r} \langle -\alpha + n_1 + \cdots + n_r \rangle
\end{aligned}
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Example

$I=\int_0^\infty e^{-x}dx=1$

$$I=\int_0^\infty \sum_n \phi_n x^n dx=\sum_n \phi_n \langle n+1\rangle=\Gamma(-(-1))=1.$$

On the other hand

$$e^{-x}=e^{-\frac{x}{3}}e^{-\frac{2x}{3}}$$

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Independence of Factorization

Theorem (L. J.)

Assume that $f(x)$ admits a representation of the form

$$f(x) = \prod_{i=1}^{r} f_i(x).$$

Then, the values of the following two integrals

$$I_1 = \int_0^\infty f(x)\,dx \text{ and } I_2 = \int_0^\infty \prod_{i=1}^{r} f_i(x)\,dx,$$

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Foundamental Theorem of Calculus

Question1

$$I_1 := \int_a^b x^k dx = \frac{b^{k+1} - a^{k+1}}{k+1}.$$

Consider the change of variables

$$t = \frac{x-a}{b-x} \Rightarrow \begin{cases} x = \frac{bt+a}{t+1} & , \\ dx = \frac{b-a}{(t+1)^2} dt & . \end{cases}$$

Then,

$$I_1 = (b-a) \int_0^\infty (bt+a)^k (t+1)^{-k-2} dt \overset{MoB}{=\!=\!=} \frac{b^{k+1} - a^{k+1}}{k+1}.$$

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Again, by the change of variables that $t = \frac{x-a}{b-x}$ we have

$$\begin{aligned} I_2 &= (a-b) \int_0^{\infty} \sum_k \phi_k C(k+1)(bt+a)^k (t+1)^{-k-2}\,dt \\ &= \cdots \\ &= \sum_k \phi_{k+1} C(k+1)\left(b^{k+1} - a^{k+1}\right) \\ &= [f(b) - f(0)] - [f(a) - f(0)] \\ &= f(b) - f(a). \end{aligned}$$

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Multi-dim Integrals

$$I = \int_{\mathbb{R}^m} f\left(x_1^2 + \cdots + x_m^2\right) dx_1 \cdots dx_m$$

(I) Usual method: Spherical Coordinate $r = x_1^2 + \cdots + x_m^2$

$$I = 2\pi^{\frac{m}{2}} \left[\int_0^\infty r^{m-1} f\left(r^2\right) dr\right] \frac{1}{\Gamma\left(\frac{m}{2}\right)}.$$

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$$f(t) = \sum_{l=0}^{\infty} \phi_l a(l) t^l \Rightarrow \int_0^\infty r^{m-1} f\left(r^2\right) dr \overset{MoB}{==} \frac{1}{2} a\left(-\frac{m}{2}\right) \Gamma\left(\frac{m}{2}\right).$$

On the other hand

$$I = 2^m \int_{\mathbb{R}_+^m} \left[\sum_{l=0}^{\infty} \phi_l a(l) \left(x_1^2 + \cdots + x_m^2\right)^l\right] dx_1 \cdots dx_m \overset{MoB}{==} \pi^{\frac{m}{2}} a\left(-\frac{m}{2}\right)$$

Example

$$I = \int_{\mathbb{R}_+^m} \frac{x_1^{p_1-1} \cdots x_m^{p_m-1}}{\left(r_0 + r_1 x_1 + \cdots + r_m x_m\right)^s} dx_1 \cdots dx_m = \frac{\Gamma(p_1) \cdots \Gamma(p_m) \Gamma(s - p_1 - p_2 - \cdots - p_n)}{r_1^{p_1} \cdots r_m^{p_m} r_0^{s-p_1-\cdots-p_m} \Gamma(s)}.$$

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Namely, for $(-km)_m$

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Especially, when $k=1$, the ratio is $\frac{1}{2}$.

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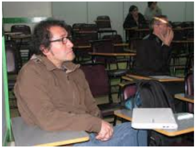
Karen Kohl–Sage+Mathematica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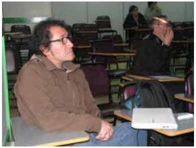Ivan Gonzalez–Maple

E_3

E_3: The value of a multi-dimensional bracket series of positive index is obtained by computing all the contributions of maximal rank by Rule E_2. These contributions to the integral appear as series in the free parameters. SERIES CONVERGING IN A COMMON REGION ARE ADDED and divergent series are discarded. Any series producing a non-real contribution is also discarded.

$$I := \int_0^\infty x J_0(xy) \frac{dx}{\sqrt{a^2+x^2}} = y^{-1} e^{-ay} = \frac{1}{y}\cosh(ay) - \frac{\sinh(ay)}{y}$$

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$I = \int_0^\infty e^{-x} dx = 1$

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Assume that $f(x)$ admits a representation of the form

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Then, the values of the following two integrals

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Bessel-K function

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$$I_\nu(x)=\sum_{m=0}^{\infty}\frac{1}{m!\Gamma(m+\nu+1)}\left(\frac{x}{2}\right)^{2m+\nu}\text{ and }K_\nu(x)=\frac{\pi}{2}\frac{I_{-\nu}(x)-I_\nu(x)}{\sin(\nu x)}.$$

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Divergent Series

Mellin Transform of K_0

$$\mathcal{M}\left(K_0\right)(s)=\int_0^{\infty}x^{s-1}K_0\left(x\right)dx=2^{s-2}\Gamma\left(\frac{s}{2}\right)^2.$$

$$\begin{aligned}
&\mathcal{M}\left(K_0\right)(s)\\
=&\int_0^{\infty}x^{s-1}K_0\left(x\right)dx\\
=&\int_0^{\infty}x^{s-1}\sum_n\phi_n\frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma\left(-n\right)}\cdot\frac{4^n}{x^{2n+1}}dx\\
=&\sum_n\phi_n\frac{\Gamma\left(n+\frac{1}{2}\right)^2 4^n}{\Gamma\left(-n\right)}\left\langle s-2n-1\right\rangle\\
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&\mathcal{M}\left(K_0\right)(s)\\
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=&\int_0^{\infty}x^{s-1}\frac{1}{2}\sum_n\phi_n\Gamma\left(-n\right)\frac{x^{2n}}{4^n}dx\\
=&\frac{1}{2}\sum_n\phi_n\frac{\Gamma\left(-n\right)}{4^n}\left\langle 2n+s\right\rangle\\
=&\frac{1}{2}\cdot\frac{1}{|2|}\cdot\frac{\Gamma\left(\frac{s}{2}\right)}{4^{-\frac{s}{2}}}\cdot\Gamma\left(\frac{s}{2}\right)\\
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Divergent Series

Mellin Transform of K_0

$$\mathcal{M}\left(K_0\right)(s)=\int_0^{\infty}x^{s-1}K_0(x)\,dx=2^{s-2}\Gamma\left(\frac{s}{2}\right)^2.$$

$$\begin{aligned}
&\mathcal{M}\left(K_0\right)(s)\\
=&\int_0^{\infty}x^{s-1}K_0(x)\,dx\\
=&\int_0^{\infty}x^{s-1}\sum_n\phi_n\frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(-n)}\cdot\frac{4^n}{x^{2n+1}}dx\\
=&\sum_n\phi_n\frac{\Gamma\left(n+\frac{1}{2}\right)^2 4^n}{\Gamma(-n)}\langle s-2n-1\rangle\\
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Divergent Series

Mellin Transform of K_0

$$\mathcal{M}(K_0)(s) = \int_0^\infty x^{s-1} K_0(x)\,dx = 2^{s-2}\Gamma\left(\frac{s}{2}\right)^2.$$

$$\begin{aligned}
&\mathcal{M}(K_0)(s)\\
=\ & \int_0^\infty x^{s-1} K_0(x)\,dx\\
=\ & \int_0^\infty x^{s-1} \sum_n \phi_n \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(-n)} \cdot \frac{4^n}{x^{2n+1}}\,dx\\
=\ & \sum_n \phi_n \frac{\Gamma\left(n+\frac{1}{2}\right)^2 4^n}{\Gamma(-n)} \langle s-2n-1\rangle\\
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&\mathcal{M}(K_0)(s)\\
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Divergent Series

Given a function $f(x)$ and its Mellin transform $\mathcal{M}(f)(s)$. We could assume f admits a series representation that

$$f(x) = \sum_n \phi_n C(n) x^{\alpha n+\beta},$$

for some $\alpha \neq 0$ and β. Applying the method of brackets yields

$$\begin{aligned} \mathcal{M}(f)(s) &= \int_0^\infty x^{s-1} f(x)\, dx \\ &\overset{P_1}{=} \sum_n \phi(n) C(n) \langle \alpha n + \beta + s \rangle \\ &\overset{E_1}{=} \frac{1}{|\alpha|} C\left(-\frac{\beta+s}{\alpha}\right) \Gamma\left(\frac{\beta+s}{\alpha}\right), \end{aligned}$$

which implies

$$C\left(-\frac{\beta+s}{\alpha}\right) = \frac{|\alpha| \mathcal{M}(f)(s)}{\Gamma\left(\frac{\beta+s}{\alpha}\right)},$$

and therefore

$$C(n) = \frac{|\alpha| \mathcal{M}(f)(-\alpha n - \beta)}{\Gamma(-n)}.$$

Product of Two Functions

Assume that in the process of evaluation of the integral

$$I = \int_0^\infty f_1(x) f_2(x)\,dx.$$

We know an expansion of $f_1(x)$ in the form

$$f_1(x) = \sum_{k=0}^{\infty} \phi_k A(k)\, x^{\alpha_1 k + \beta_1},$$

and the Mellin transform of the function $f_2(x)$

$$\mathcal{M}(s) = \mathcal{M}(f_2)(s) = \int_0^\infty x^{s-1} f_2(x)\,dx.$$

$$f_2(x) = \sum_{n=0}^{\infty} \phi_n \frac{|\alpha_2|\, \mathcal{M}(-\alpha_2 n - \beta_2)}{\Gamma(-n)} x^{\alpha_2 n + \beta_2},$$

which further leads to

$$I = \sum_{k,n} \phi_{k,n} \frac{|\alpha_2|\, A(k)\, \mathcal{M}(-\alpha_2 n - \beta_2)}{\Gamma(-n)} \langle \alpha_1 k + \alpha_2 n + \beta_1 + \beta_2 + 1 \rangle.$$

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Product of Two Functions

THM. [I. Gonzalez, L. J. V. H. Moll]

$$I = \int_0^\infty f_1(x) f_2(x)\, dx$$

$$= \begin{cases} \sum_k \phi_k A(k) \mathcal{M}(\alpha_1 k + \beta_1 + 1) \\ \left|\frac{\alpha_2}{\alpha_1}\right| \sum_n \frac{\phi_n A\left(-\frac{\alpha_2 n + \beta_1 + \beta_2 + 1}{\alpha_1}\right) \mathcal{M}(-\alpha_2 n - \beta_2) \Gamma\left(\frac{\alpha_2 n + \beta_1 + \beta_2 + 1}{\alpha_1}\right)}{\Gamma(-n)} \end{cases}$$

Product of Two Functions

$I=\int_0^\infty J_\mu(ax)\,J_\nu(bx)\,dx$

$$\begin{cases} f_1=J_\mu(ax)=\sum_{k=0}^{\infty}\frac{\phi_k a^{2k+\mu}}{\Gamma(k+\mu+1)2^{2k+\mu}}x^{2k+\mu} \\ \mathcal{M}(f_2)(s)=\mathcal{M}(J_\nu(bx))(s)=\int_0^\infty x^{s-1}J_\nu(x)\,dx=\frac{2^{s-1}\Gamma\left(\frac{\nu+s}{2}\right)}{b^s\Gamma\left(1+\frac{\nu-s}{2}\right)} \end{cases}$$

Then,

$$I=\sum_{k=0}^{\infty}\phi_k A(k)\,\mathcal{M}(\alpha_1 k+\beta+1)=a^\mu b^{-\mu-1}\frac{\Gamma\left(\frac{\mu+\nu+1}{2}\right)}{\Gamma(\mu+1)\,\Gamma\left(\frac{\nu-\mu+1}{2}\right)},$$

if $b>a$.

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$I=\int_0^\infty J_\mu(ax)\,J_\nu(bx)\,dx$

And also,

$$I=\frac{|\alpha_2|}{2}\sum_{n=0}^{\infty}\frac{\phi_n}{\Gamma(-n)}A\left(-\frac{\alpha_2 n+\mu+\beta_2+1}{2}\right)\mathcal{M}(-\alpha_2 n-\beta_2)\,\Gamma\left(\frac{\alpha_2 n+\mu+\beta_2+1}{2}\right).$$

Either, we know $\alpha_2=2$ and $\beta_2=\nu$ or, we choose them so that $\Gamma\left(\frac{\alpha_2 n+\mu+\beta_2+1}{2}\right)$ can cancel $\Gamma(-n)$. Then

$$\begin{aligned}I &= \sum_{n=0}^{\infty}\frac{\phi_n}{\Gamma(-n)}A\left(-n-\frac{\mu+\nu+1}{2}\right)\mathcal{M}(-2n-\nu)\,\Gamma\left(n+\frac{\mu+\nu+1}{2}\right)\\ &= b^\nu a^{-\nu-1}\sum_{n=0}^{\infty}\frac{\phi_n\Gamma\left(n+\frac{\mu+\nu+1}{2}\right)}{\Gamma(n+\nu+1)\,\Gamma\left(-n+\frac{\mu-\nu+1}{2}\right)}\cdot\left(\frac{b^2}{a^2}\right)^n\\ &= b^\nu a^{-\nu-1}\frac{\Gamma\left(\frac{\mu+\nu+1}{2}\right)}{\Gamma(\nu+1)\,\Gamma\left(\frac{\mu-\nu+1}{2}\right)}\,{}_2F_1\left(\left.\begin{matrix}\frac{\mu+\nu+1}{2},\frac{\nu-\mu+1}{2}\\ \nu+1\end{matrix}\right|\frac{b^2}{a^2}\right),\quad a>b\end{aligned}$$

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Negative Dimension

Idea:

$$\int_{\mathbb{R}^D} e^{-\alpha \mathbf{x}^2} d^D \mathbf{x} = \left(\frac{\pi}{\alpha}\right)^{\frac{D}{2}},$$

where

$$\begin{cases} \mathbb{R}^D & = \{x_1, x_2, \dots, x_D\}, \\ \mathbf{x}^2 & = x_1^2 + x_2^2 + \cdots + x_D^2, \\ d^D \mathbf{x} & = dx_1 dx_2 \cdots dx_D. \end{cases}$$

Example

$$I = \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$$

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$$e^{-\alpha} \sqrt{\frac{\pi}{\alpha}} = \sqrt{\pi} \sum_{n=0}^{\infty} \phi_n \alpha^{n-\frac{1}{2}};$$

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Matching $[\alpha]$ gives $m = n - \frac{1}{2}$ and by AC,

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Negative Dimension

$I = \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$

$$\int_0^\infty \frac{1}{(1+x^2)^n} dx \overset{P_2}{=} \int_0^\infty \sum_{k,l} \phi_{k,l} x^{2l} \frac{\langle n+k+l \rangle}{\Gamma(n)} dx$$

$$\overset{P_1}{=} \frac{1}{\Gamma(n)} \sum_{k,l} \phi_{k,l} \langle n+k+l \rangle \langle 2l+1 \rangle$$

$$\overset{E_2}{=} \frac{1}{\Gamma(n)} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \Gamma\left(n - \frac{1}{2}\right)$$

$$= \frac{\sqrt{\pi}\,\Gamma\left(-\frac{1}{2}+n\right)}{2\Gamma(n)}$$

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$$\begin{aligned}
\int_0^\infty \frac{1}{(1+x^2)^n} dx &\overset{P_2}{=} \int_0^\infty \sum_{k,l} \phi_{k,l} x^{2l} \frac{\langle n+k+l \rangle}{\Gamma(n)} dx \\
&\overset{P_1}{=} \frac{1}{\Gamma(n)} \sum_{k,l} \phi_{k,l} \langle n+k+l \rangle \langle 2l+1 \rangle \\
&\overset{E_2}{=} \frac{1}{\Gamma(n)} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \Gamma\left(n - \frac{1}{2}\right) \\
&= \frac{\sqrt{\pi}\,\Gamma\left(-\frac{1}{2}+n\right)}{2\Gamma(n)}
\end{aligned}$$

Integration by Differentiation

$$
\begin{aligned}
\int_a^b f(x)\,dx &= \lim_{\varepsilon\to 0} f(\partial_\varepsilon)\frac{e^{\varepsilon b}-e^{\varepsilon a}}{\varepsilon}, \\
\int_{-\infty}^{\infty} f(x)\,dx &= \lim_{\varepsilon\to 0} 2\pi f(-\iota\partial_\varepsilon)\,\delta(\varepsilon) = 2\pi\delta(\iota\partial_\varepsilon)\,f(\varepsilon), \\
\int_0^{\infty} f(x)\,dx &= \lim_{\varepsilon\to 0} f(-\partial_\varepsilon)\frac{1}{\varepsilon}, \\
\int_{-\infty}^{0} f(x)\,dx &= \lim_{\varepsilon\to 0} f(\partial_\varepsilon)\frac{1}{\varepsilon}, \\
\int_{-\infty}^{\infty} f(x)\,dx &= \lim_{\varepsilon\to 0} \left[f(-\partial_\varepsilon)+f(\partial_\varepsilon)\right]\frac{1}{\varepsilon},
\end{aligned}
$$

where $\partial_\varepsilon = \frac{\partial}{\partial\varepsilon}$.

Integration by Differentiation

$I=\int_0^\infty \frac{\sin x}{x}=\frac{\pi}{2}$

$$f\left(x\right)=\frac{1}{x}\cdot\frac{1}{2\iota}\left(e^{\iota x}-e^{-\iota x}\right)$$

$$I=\lim_{\varepsilon\to0}f\left(-\partial_{\varepsilon}\right)\frac{1}{\varepsilon}=\frac{1}{2\iota}\lim_{\varepsilon\to0}\left(e^{-\iota\partial_{\varepsilon}}-e^{\iota\partial_{\varepsilon}}\right)\frac{1}{\partial_{\varepsilon}}\circ\frac{1}{\epsilon}.$$

Note that

$$\frac{1}{\partial_{\varepsilon}}\circ\frac{1}{\epsilon}=\int\frac{1}{\varepsilon}d\varepsilon=\ln\varepsilon+c,$$

and

$$e^{a\partial_x}\circ g\left(x\right)=g\left(x+a\right).$$

Therefore,

$$I=\frac{1}{2\iota}\lim_{\varepsilon\to0}\left[\left(\ln\left(\varepsilon-\iota\right)+c\right)-\left(\ln\left(\varepsilon+\iota\right)+c\right)\right]=\frac{\pi}{2}.$$

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Thank You!