The Method of Brackets (MoB)

Lin Jiu

RESEARCH INSTITUTE FOR SYMBOLIC COMPUTATION | RISC JKU JOHANNES KEPLER UNIVERSITÄT LINZ

The 5^{th} International Congress on Mathematical Software
@ Zuse Institute Berlin

July 13^{th} 2016

Acknowledgement

Joint Work with:

Victor Hugo Moll

Karen Kohl

Ivan Gonzalez

Outlines

1. Acknowledgement & Outlines
2. Introduction
   - Rules
   - Ramanujan's Master Theorem (RMT)
   - Examples
3. Work
   - Factorization of the Integrand
   - Implementation
   - Future Work

Rules

Idea

MoB evaluates the definite integral

$$\int_0^\infty f(x)\,dx$$

(most of the time) in terms of SERIES, with *ONLY SIX* rules:

Defintion [Indicator]

$$\phi_n := \frac{(-1)^n}{n!} = \frac{(-1)^n}{\Gamma(n+1)}$$

and

$$\phi_{1,\ldots,r} := \phi_{n_1,\ldots,n_r} = \phi_{n_1}\phi_{n_2}\cdots\phi_{n_r} = \prod_{i=1}^{r}\phi_{n_i}.$$

Rules (P-Production; E-Evaluation) $I=\int_0^\infty f(x)\,dx$

P_1: $f(x)=\sum_{n=0}^{\infty}a_nx^{\alpha n+\beta-1}\Rightarrow\int_0^\infty f(x)\,dx\mapsto\sum_n a_n\langle\alpha n+\beta\rangle$—Bracket Series;

P_2: For $\alpha<0$, $(a_1+\cdots+a_r)^\alpha\mapsto\sum_{n_1,\ldots,n_r}\phi_{1,\ldots,r}a_1^{n_1}\cdots a_r^{n_r}\frac{\langle-\alpha+n_1+\cdots+n_r\rangle}{\Gamma(-\alpha)}$;

P_3: For each bracket series, we assign *index=# of sums− # of brackets*;

E_1: $\sum_n\phi_nf(n)\langle\alpha n+\beta\rangle=\frac{1}{|\alpha|}f(n^*)\Gamma(-n^*)$, where n^* solves $\alpha n+\beta=0$;

E_2:
$$\sum_{n_1,\ldots,n_r}\phi_{1,\ldots,r}f(n_1,\ldots,n_r)\prod_{i=1}^{r}\langle a_{i1}n_1+\cdots+a_{ir}n_r+c_i\rangle=\frac{f(n_1^*,\ldots,n_r^*)\prod_{i=1}^{r}\Gamma(-n_i^*)}{|\det A|},$$

$$(n_1^*,\ldots,n_r^*)\ \text{solves}\ \begin{cases}a_{11}n_1+\cdots+a_{1r}n_r+c_1 & =0\\ \ldots & \cdots\\ a_{r1}n_1+\cdots+a_{rr}n_r+c_r & =0\end{cases};$$

E_3: The value of a multi-dimensional bracket series of ***POSITIVE*** index is obtained by computing all the contributions of maximal rank by Rule E_2. These contributions to the integral appear as series in the free parameters. Series converging in a common region are added and divergent series are discarded. Any series producing a non-real contribution is also discarded.

Ramanujan's Master Theorem[RMT]

Theorem[RMT]

$$\int_0^\infty x^{s-1}\left\{a(0)-\frac{a(1)}{1!}x+\frac{a(2)}{2!}x^2-\cdots\right\}dx=a(-s)\Gamma(s)$$

(1)

$$\int_0^\infty x^{s-1}\left(\sum_{n=0}^\infty \phi_n a(n)x^n\right)dx=a(-s)\Gamma(s)$$

(2) [Hardy]

•$H(\delta):=\{s=\sigma+\iota t:\sigma\geq-\delta,0<\delta<1\}$;

•$\psi(x)\in C^\infty(H(\delta))$; $\exists C,P,A,\ A<\pi$ such that $|\psi(s)|\leq Ce^{P\delta+A|t|}$, $\forall s\in H(\delta)$;

•$0<c<\delta,\ \Psi(x):=\frac{1}{2\pi\iota}\int_{c-\iota\infty}^{c+\iota\infty}\frac{\pi}{\sin(\pi s)}\psi(-s)x^{-s}dx \overset{0<x<e^{-P}}{=\!=\!=\!=} \sum_{k=0}^\infty\psi(k)(-x)^k$;

$$\int_0^\infty \Psi(x)x^{s-1}dx=\frac{\pi}{\sin(\pi s)}\psi(-s).$$

Rules Again

Theorem[RMT]

$$\int_0^{\infty} x^{s-1}\left\{\sum_{n=0}^{\infty}\phi_n a(n) x^n\right\} dx = a(-s)\Gamma(s)$$

(1) Integrand$\mapsto$Power Series;

(2) Keep Track of s;

(3) Apply the Formula;

(4) Multiple Integrals;

$$\int_0^{\infty}\int_0^{\infty}\sum_{n,m} a(m,n) x^m y^n dxdy = ?$$

(5) More Sums than Integrals (brackets);

$$\int_0^{\infty} f_1(x) f_2(x) dx = \int_0^{\infty}\sum_{m,n} a(m,n) x^{m+n} dx = \sum_{m,n} a(m,n)\langle m+n+1\rangle = ?$$

(6) Extra.

Rules Again

P_1: $f(x)=\sum_{n=0}^{\infty} a_n x^{\alpha n+\beta-1} \Rightarrow \int_0^\infty f(x)\,dx \mapsto \sum_n a_n \langle \alpha n+\beta\rangle$ $\boxed{s-1\mapsto s}$

P_2: For $\alpha<0$,$(a_1+\cdots+a_r)^\alpha \mapsto \sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} a_1^{n_1}\cdots a_r^{n_r} \frac{\langle -\alpha+n_1+\cdots+n_r\rangle}{\Gamma(-\alpha)}$;

P_3: Index=# of sums− # of brackets; Just a definition

E_1: $\sum_n \phi_n f(n)\langle \alpha n+\beta\rangle = \frac{f(n^*)\Gamma(-n^*)}{|\alpha|}$,, n^* solves $\alpha n+\beta=0$; RMT

E_2: Iteration of RMT

$$\sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} f(n_1,\ldots,n_r) \prod_{i=1}^{r} \langle a_{i1}n_1+\cdots+a_{ir}n_r+c_i\rangle = \frac{f(n_1^*,\ldots,n_r^*)\prod_{i=1}^{r}\Gamma(-n_i^*)}{|\det A|}$$

E_3: The value of a multi-dimensional bracket series of ***POSITIVE*** index is obtained by computing all the contributions of maximal rank by Rule E_2. These contributions to the integral appear as series in the free parameters. Series converging in a common region are added and divergent series are discarded. Any series producing a non-real contribution is also discarded.

Rules Again

Theorem[RMT]

$$\int_0^\infty x^{s-1}\left\{\sum_{n=0}^{\infty}\phi_n a(n)x^n\right\}dx = a(-s)\Gamma(s)$$

(1) Integrand$\mapsto$Power Series;
(2) Keep Track of s;
(3) Apply the Formula;
(4) Multiple Integrals;

$$\int_0^\infty\int_0^\infty\sum_{n,m}a(m,n)x^m y^n dxdy = ?$$

(5) More Sums than Integrals (brackets);

$$\int_0^\infty f_1(x)f_2(x)dx = \int_0^\infty\sum_{m,n}a(m,n)x^{m+n}dx = \sum_{m,n}a(m,n)\langle m+n+1\rangle = ?$$

(6) Extra.

Rules Again

P_1: $f(x) = \sum_{n=0}^{\infty} a_n x^{\alpha n+\beta-1} \Rightarrow \int_0^\infty f(x)\,dx \mapsto \sum_n a_n \langle \alpha n + \beta \rangle$ $\boxed{s-1 \mapsto s}$

P_2:For $\alpha < 0$, $(a_1 + \cdots + a_r)^\alpha \mapsto \sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} a_1^{n_1} \cdots a_r^{n_r} \frac{\langle -\alpha + n_1 + \cdots + n_r \rangle}{\Gamma(-\alpha)}$;

P_3: Index=# of sums− # of brackets; $\boxed{\text{Just a definition}}$

E_1: $\sum_n \phi_n f(n) \langle \alpha n + \beta \rangle = \frac{f(n^*)\Gamma(-n^*)}{|\alpha|}$, n^* solves $\alpha n + \beta = 0$; $\boxed{\text{RMT}}$

E_2: $\boxed{\text{Iteration of RMT}}$

$$\sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} f(n_1,\ldots,n_r) \prod_{i=1}^{r} \langle a_{i1}n_1 + \cdots + a_{ir}n_r + c_i \rangle = \frac{f(n_1^*,\ldots,n_r^*) \prod_{i=1}^{r} \Gamma(-n_i^*)}{|\det A|}$$

E_3: The value of a multi-dimensional bracket series of ***POSITIVE*** index is obtained by computing all the contributions of maximal rank by Rule E_2. These contributions to the integral appear as series in the free parameters. Series converging in a common region are added and divergent series are discarded. Any series producing a non-real contribution is also discarded.

Rules Again

Theorem[RMT]

$$\int_0^\infty x^{s-1}\left\{\sum_{n=0}^\infty \phi_n a(n)\, x^n\right\} dx = a(-s)\,\Gamma(s)$$

(1) Integrand$\mapsto$Power Series;
(2) Keep Track of s;
(3) Apply the Formula;
(4) Multiple Integrals;

$$\int_0^\infty \int_0^\infty \sum_{n,m} a(m,n)\, x^m y^n dxdy = ?$$

(5) More Sums than Integrals (brackets);

$$\int_0^\infty f_1(x)\, f_2(x)\, dx = \int_0^\infty \sum_{m,n} a(m,n)\, x^{m+n} dx = \sum_{m,n} a(m,n)\, \langle m+n+1\rangle = ?$$

(6) Extra.

Rules Again

P_1: $f(x) = \sum_{n=0}^{\infty} a_n x^{\alpha n + \beta - 1} \Rightarrow \int_0^{\infty} f(x)\, dx \mapsto \sum_n a_n \langle \alpha n + \beta \rangle$ $\boxed{s - 1 \mapsto s}$

P_2:For $\alpha < 0$, $(a_1 + \cdots + a_r)^{\alpha} \mapsto \sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} a_1^{n_1} \cdots a_r^{n_r} \frac{\langle -\alpha + n_1 + \cdots + n_r \rangle}{\Gamma(-\alpha)}$;

P_3: Index=# of sums− # of brackets; $\boxed{\text{Just a definition}}$

E_1: $\sum_n \phi_n f(n) \langle \alpha n + \beta \rangle = \frac{f(n^*)\Gamma(-n^*)}{|\alpha|}$, n^* solves $\alpha n + \beta = 0$; $\boxed{\text{RMT}}$

E_2: $\boxed{\text{Iteration of RMT}}$

$$\sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} f(n_1,\ldots,n_r) \prod_{i=1}^{r} \langle a_{i1} n_1 + \cdots + a_{ir} n_r + c_i \rangle = \frac{f(n_1^*,\ldots,n_r^*) \prod_{i=1}^{r} \Gamma(-n_i^*)}{|\det A|}$$

E_3: The value of a multi-dimensional bracket series of ***POSITIVE*** index is obtained by computing all the contributions of maximal rank by Rule E_2. These contributions to the integral appear as series in the free parameters. Series converging in a common region are added and divergent series are discarded. Any series producing a non-real contribution is also discarded.

Rules Again

Theorem[RMT]

$$\int_0^\infty x^{s-1}\left\{\sum_{n=0}^\infty \phi_n a(n) x^n\right\} dx = a(-s)\Gamma(s)$$

(1) Integrand$\mapsto$Power Series;
(2) Keep Track of s;
(3) Apply the Formula;
(4) Multiple Integrals;

$$\int_0^\infty \int_0^\infty \sum_{n,m} a(m,n) x^m y^n dxdy = ?$$

(5) More Sums than Integrals (brackets);

$$\int_0^\infty f_1(x) f_2(x) dx = \int_0^\infty \sum_{m,n} a(m,n) x^{m+n} dx = \sum_{m,n} a(m,n)\langle m+n+1\rangle = ?$$

(6) Extra.

Rules Again

P_1: $f(x)=\sum_{n=0}^{\infty} a_n x^{\alpha n+\beta-1} \Rightarrow \int_0^\infty f(x)\,dx \mapsto \sum_n a_n \langle \alpha n+\beta\rangle$ $\boxed{s-1\mapsto s}$

P_2:For $\alpha<0$, $(a_1+\cdots+a_r)^\alpha \mapsto \sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} a_1^{n_1}\cdots a_r^{n_r}\frac{\langle -\alpha+n_1+\cdots+n_r\rangle}{\Gamma(-\alpha)}$;

P_3: Index=# of sums− # of brackets; Just a definition

E_1: $\sum_n \phi_n f(n)\langle \alpha n+\beta\rangle=\frac{f(n^*)\Gamma(-n^*)}{|\alpha|}$, n^* solves $\alpha n+\beta=0$; RMT

E_2: Iteration of RMT

$$\sum_{n_1,\ldots,n_r}\phi_{1,\ldots,r} f(n_1,\ldots,n_r)\prod_{i=1}^{r}\langle a_{i1}n_1+\cdots+a_{ir}n_r+c_i\rangle=\frac{f(n_1^*,\ldots,n_r^*)\prod_{i=1}^{r}\Gamma(-n_i^*)}{|\det A|}$$

E_3: The value of a multi-dimensional bracket series of ***POSITIVE*** index is obtained by computing all the contributions of maximal rank by Rule E_2. These contributions to the integral appear as series in the free parameters. Series converging in a common region are added and divergent series are discarded. Any series producing a non-real contribution is also discarded.

Rules Again

P_1: $f(x)=\sum_{n=0}^{\infty} a_n x^{\alpha n+\beta-1} \Rightarrow \int_0^\infty f(x)\,dx \mapsto \sum_n a_n \langle \alpha n+\beta\rangle$ $\boxed{s-1\mapsto s}$

P_2:For $\alpha<0$, $(a_1+\cdots+a_r)^\alpha \mapsto \sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} a_1^{n_1}\cdots a_r^{n_r} \frac{\langle -\alpha+n_1+\cdots+n_r\rangle}{\Gamma(-\alpha)}$;

P_3: Index=# of sums− # of brackets; Just a definition

E_1: $\sum_n \phi_n f(n)\langle \alpha n+\beta\rangle = \frac{f(n^*)\Gamma(-n^*)}{|\alpha|}$, n^* solves $\alpha n+\beta=0$; RMT

E_2: Iteration of RMT

$$\sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} f(n_1,\ldots,n_r)\prod_{i=1}^{r}\langle a_{i1}n_1+\cdots+a_{ir}n_r+c_i\rangle = \frac{f(n_1^*,\ldots,n_r^*)\prod_{i=1}^{r}\Gamma(-n_i^*)}{|\det A|}$$

E_3: The value of a multi-dimensional bracket series of ***POSITIVE*** index is obtained by computing ***all the contributions*** of maximal rank by Rule E_2. These contributions to the integral appear as series in the free parameters. Series converging in a common region are added and divergent series are discarded. Any series producing a non-real contribution is also discarded.

Rules Again

Theorem[RMT]

$$\int_0^{\infty} x^{s-1} \left\{ \sum_{n=0}^{\infty} \phi_n a(n) x^n \right\} dx = a(-s) \Gamma(s)$$

(1) Integrand$\mapsto$Power Series;
(2) Keep Track of s;
(3) Apply the Formula;
(4) Multiple Integrals;

$$\int_0^{\infty} \int_0^{\infty} \sum_{n,m} a(m,n) x^m y^n dxdy =?$$

(5) More Sums than Integrals (brackets);

$$\int_0^{\infty} f_1(x) f_2(x) dx = \int_0^{\infty} \sum_{m,n} a(m,n) x^{m+n} dx = \sum_{m,n} a(m,n) \langle m+n+1 \rangle =?$$

(6) Extra.

Rules Again

P_1: $f(x) = \sum_{n=0}^{\infty} a_n x^{\alpha n+\beta-1} \Rightarrow \int_0^{\infty} f(x)\,dx \mapsto \sum_n a_n \langle \alpha n + \beta \rangle$ $\boxed{s-1 \mapsto s}$

P_2:For $\alpha < 0$, $(a_1 + \cdots + a_r)^{\alpha} \mapsto \sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} a_1^{n_1} \cdots a_r^{n_r} \frac{\langle -\alpha + n_1 + \cdots + n_r \rangle}{\Gamma(-\alpha)}$;

P_3: Index=# of sums− # of brackets; Just a definition

E_1: $\sum_n \phi_n f(n) \langle \alpha n + \beta \rangle = \frac{f(n^*)\Gamma(-n^*)}{|\alpha|}$, n^* solves $\alpha n + \beta = 0$; RMT

E_2: Iteration of RMT

$$\sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} f(n_1,\ldots,n_r) \prod_{i=1}^{r} \langle a_{i1} n_1 + \cdots + a_{ir} n_r + c_i \rangle = \frac{f(n_1^*,\ldots,n_r^*) \prod_{i=1}^{r} \Gamma(-n_i^*)}{|\det A|}$$

E_3: The value of a multi-dimensional bracket series of ***POSITIVE*** index is obtained by computing all the contributions of maximal rank by Rule E_2. These contributions to the integral appear as series in the free parameters. Series converging in a common region are added and divergent series are discarded. Any series producing a non-real contribution is also discarded.

Rule P_2

$$
\begin{aligned}
& \frac{\Gamma(-\alpha)}{(a_1+\cdots+a_r)^{-\alpha}}. \\
= & \int_0^\infty x^{-\alpha-1} e^{-(a_1+\cdots+a_r)x} dx \\
= & \int_0^\infty x^{-\alpha-1} e^{-a_1 x} e^{-a_2 x} \cdots e^{-a_r x} dx \\
= & \int_0^\infty x^{-\alpha-1} \prod_{i=1}^{r} \left(\sum_{n_i=0}^{\infty} \phi_{n_i} (ax)^{n_i} \right) dx \\
= & \int_0^\infty \sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} a_1^{n_1} \cdots a_r^{n_r} x^{n_1+\cdots+n_r-\alpha-1} dx \\
= & \sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} a_1^{n_1} \cdots a_r^{n_r} \langle -\alpha + n_1 + \cdots + n_r \rangle
\end{aligned}
$$

Examples

$$I := \int_0^\infty x J_0(xy) \frac{dx}{\sqrt{a^2+x^2}} = y^{-1} e^{-ay} \quad [y>0 \ Re(a)>0]$$

Rule P_2:

$$\frac{1}{\sqrt{a^2+x^2}} = \left(a^2+x^2\right)^{-\frac{1}{2}} = \sum_{n_1,n_2} \phi_{1,2} a^{2n_1} x^{2n_2} \frac{\left\langle \frac{1}{2}+n_1+n_2 \right\rangle}{\Gamma\left(\frac{1}{2}\right)}$$

$J_0(xy)$

$$J_0(xy) = \sum_{n_3} \phi_{n_3} \frac{y^{2n_3}}{\Gamma(n_3+1)\, 2^{2n_3}} x^{2n_3}$$

Rule P_1

$$\begin{aligned} I &= \int_0^\infty \sum_{n_1,n_2,n_3} \phi_{1,2,3} \frac{y^{2n_3} a^{2n_1}}{\Gamma(n_3+1)\Gamma\left(\frac{1}{2}\right) 2^{2n_3}} \left\langle n_1+n_2+\frac{1}{2} \right\rangle x^{2n_2+2n_3+1} dx \\ &= \sum_{n_1,n_2,n_3} \phi_{1,2,3} \frac{y^{2n_3} a^{2n_1}}{\Gamma(n_3+1)\Gamma\left(\frac{1}{2}\right) 2^{2n_3}} \left\langle n_1+n_2+\frac{1}{2} \right\rangle \langle 2n_2+2n_3+2 \rangle \end{aligned}$$

Examples

$$I := \int_0^\infty xJ_0(xy)\frac{dx}{\sqrt{a^2+x^2}} = y^{-1}e^{-ay}$$

$$I = \sum_{n_1,n_2,n_3} \phi_{1,2,3} \frac{y^{2n_3}a^{2n_1}}{\Gamma(n_3+1)\Gamma\left(\frac{1}{2}\right)2^{2n_3}} \left\langle n_1+n_2+\frac{1}{2}\right\rangle \langle 2n_2+2n_3+2\rangle;$$

n_1 free: $n_2^* = -\frac{1}{2} - n_1$; $n_3^* = -\frac{1}{2} + n_1$; $\det = 2$:

$$\begin{aligned} I &= \frac{1}{2}\sum_{n_1}\phi_{n_1}\frac{y^{2n_1-1}a^{2n_1}}{\Gamma\left(n_1+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)2^{2n_1-1}}\Gamma\left(n_1+\frac{1}{2}\right)\Gamma\left(-n_1+\frac{1}{2}\right) \\ &= \frac{1}{y}\sum_{n_1=0}^{\infty}\phi_{n_1}\left(\frac{ay}{2}\right)^{2n_1}\frac{\Gamma\left(\frac{1}{2}-n_1\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{1}{y}\cosh(ay); \end{aligned}$$

n_2 free : $I = \frac{1}{\sqrt{\pi}y}\sum_{n_2=0}^{\infty}\frac{\Gamma\left(n_2+\frac{1}{2}\right)}{\Gamma(-n_2)}\left(\frac{2}{ay}\right)^{2n_2+1} = 0$; n_3 free : $I = \text{Series} = -\frac{\sinh(ay)}{y}$;

E_3 : $$I = \frac{1}{y}\cosh(ay) - \frac{\sinh(ay)}{y} = y^{-1}e^{-ay}.$$

Example

$I=\int_0^\infty e^{-x}dx=1$

$$I=\int_0^\infty \sum_n \phi_n x^n dx = \sum_n \phi_n \langle n+1\rangle = \Gamma(-(-1)) = 1.$$

On the other hand

$$e^{-x}=e^{-\frac{x}{3}}e^{-\frac{2x}{3}} \quad \left(e^{-ax}e^{-bx},\ a+b=1\right)$$

$$I=\int_0^\infty \left(\sum_{n_1}\phi_{n_1}\frac{x^{n_1}}{3^{n_1}}\right)\left(\sum_{n_2}\phi_{n_2}\frac{2^{n_2}x^{n_2}}{3^{n_2}}\right)dx=\sum_{n_1,n_2}\phi_{1,2}\frac{2^{n_2}}{3^{n_1+n_2}}\langle n_1+n_2+1\rangle$$

$$I=\begin{cases} n_2^*=-1-n_1: & \sum_{n_1}\phi_{n_1}\frac{3}{2^{n_1+1}}\Gamma(n_1+1)=\frac{3}{2}\cdot\sum_{n_1}\left(-\frac{1}{2}\right)^{n_1}=1; \\ n_1^*=-1-n_2: & \sum_{n_2}\phi_{n_2}3\cdot 2^{n_2}\Gamma(n_2+1)=3\cdot\sum_{n_2}(-2)^{n_2}\stackrel{AC}{=}1. \end{cases}$$

Independence of Factorization

Theorem (L. J.)

Assume that $f(x)$ admits a representation of the form

$$f(x) = \prod_{i=1}^{r} f_i(x).$$

Then, the values of the following two integrals

$$I_1 = \int_0^\infty f(x)\,dx \text{ and } I_2 = \int_0^\infty \prod_{i=1}^{r} f_i(x)\,dx,$$

obtained by applying the Method of Brackets, are the same.

Remark

The proof uses analytic continuation of multinomial coefficients.

Implementation

Karen Kohl–***Sage+Mathematica***

Ivan Gonzalez–***Maple***

Mathematica

Implementation

$$\int_0^\infty \frac{dx}{(1+x^2)^{m+1}} = \frac{\pi}{2^{2m+1}}\binom{2m}{m}$$

Example_Definite_Integrals_By_The_Method_of_Brackets_Part1.nb * - Wolfram Mathematica 10.4 (on qftquad3)

```
brackets = MakeTheBrackets[BPower[1 + BPower[x, 2], -m - 1], {x}]
(*Example of ∫_0^∞ dx/(1+x^2)^(m+1) = π/2^(2m+1) (2m m)*)
```

Out[33]=

```
{{1, BPower[-1, n[1]], BPower[-1, n[2]],
  BPower[Gamma[1 + m], -1], BPower[Gamma[1 + n[1]], -1],
  BPower[Gamma[1 + n[2]], -1]}, {1 + m + n[1] + n[2], 1 + 2 n[2]}, 2}
```

In[34]:= `ReadTheBrackets[brackets]`

$$\frac{(-1)^{n[1]+n[2]}}{\Gamma[1+m]\,\Gamma[1+n[1]]\,\Gamma[1+n[2]]}$$

$$\begin{pmatrix} 1+m \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} n[1] \\ n[2] \end{pmatrix}$$

In[35]:= `result = EvaluateTheBrackets[brackets]`

Out[35]=

$$\Big\{\Big\{\{\},\ \frac{1}{2}\sqrt{\pi}\,\text{BPower}\Big[-1, \frac{1}{2}(-1-2m)\Big]\,\text{BPower}\Big[-1, \frac{1}{2}(1+2m)\Big]\,\text{BPower}\Big[\text{Gamma}\Big[1+\frac{1}{2}(-1-2m)\Big], -1\Big]\ \text{BPower}\Big[\text{Gamma}\Big[1+\frac{1}{2}(-1-2m)\Big], 1\Big]\,\text{BPower}[\text{Gamma}[1+m], -1]\,\text{Gamma}\Big[\frac{1}{2}(1+2m)\Big]\Big\}\Big\}$$

In[36]:= `result /. BPower → Power`

Out[36]=

$$\Big\{\Big\{\{\},\ \frac{\sqrt{\pi}\,\text{Gamma}\big[\frac{1}{2}(1+2m)\big]}{2\,\text{Gamma}[1+m]}\Big\}\Big\}$$

Other Work

- Pochhammer: [I. Gozanlez, L. J. V. H. Moll] Let $m, k \in \mathbb{N}$,

$$\lim_{\varepsilon \to 0} (-k(m+\varepsilon))_{-(m+\varepsilon)} = \frac{(-1)^m (km)!}{((k+1)m)!} \cdot \frac{k}{k+1}.$$

- Divergent Series:

$$K_0 = \int_0^\infty \frac{\cos(xt)}{\sqrt{1+t^2}} dt \overset{\text{MoB}}{==} \begin{cases} \frac{1}{2} \sum_n \phi_n \Gamma(-n) \frac{x^{2n}}{4^n} \\ \sum_n \phi_n \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(-n)} \cdot \frac{4^n}{x^{2n+1}} \end{cases} \Rightarrow \mathcal{M}(K_0)(s) \overset{\text{MoB}}{==} 2^{s-2} \Gamma\left(\frac{s}{2}\right)^2.$$

- Comparison:

  (1) Negative Dimensional Integration Method;
  (2) Integration by Differentiation;

Future Work

- E_3: $\cdots$ SERIES CONVERGING IN A COMMON REGION ARE ADDED and divergent series are discarded $\cdots$

$$I := \int_0^\infty x J_0(xy) \frac{dx}{\sqrt{a^2+x^2}} = \frac{1}{y}\cosh(ay) - \frac{\sinh(ay)}{y} = y^{-1}e^{-ay}.$$

- Analytic Continuation
- RMT/Precision/Classes
- $(0,\infty) \mapsto ???$

Reference

B. C. Berndt, *Ramanujan's Notebooks Part I*, Springer-Verlag, 1991.

K. Boyadzhiev, V. H. Moll. The integrals in Gradshteyn and Ryzhik. Part 21: hyperbolic functions. *Scientia*. **22** (2013), 109–127, 2013.

I. Gonzalez, K. Kohl, and V. H. Moll. Evaluation of entries in Gradshteyn and Ryzhik employing the method of brackets. *Scientia*, **25** (2014), 65–84.

I. Gonzalez and V. H. Moll. Definite integrals by the method of brackets. Part 1. *Adv. Appl. Math.*, **45** (2010), 50–73.

I. Gonzalez and I. Schmidt. Optimized negative dimensional integration method (NDIM) and multiloop Feynman diagram calculation. *Nuclear Phys. B*, **769** (2017), 124–173.

I. Gonzalez and I. Schmidt. Modular application of an integration by fractional expansion (IBFE) method to multiloop Feynman diagrams. *Phys. Rev. D*, **78** (2008), 086003.

I. G. Halliday and R. M. Ricotta. Negative dimensional integrals. I. Feynman graphs. *Phys. Lett. B*, **193** (1987), 241–246.

I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products*, Academic Press, 2015.

G. H. Hardy. *Ramanujan. Twelve Lectures on Subjects Suggested by His Life and Work*. Chelsea Publishing Company, 1987.

K. Kohl. *Algorithmic Methods for Definite Integration*. PhD thesis, Tulane University, 2011.

End

Thank You!