Random Walk: A Probabilistic and Geometric Approach to Number Theory

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2016 International Conference on Mathematical Characterization, Analysis and Applications of Complex Information
July 19th 2016

Acknowledgement

Joint Work with:

Christophe Vignat

Victor Hugo Moll

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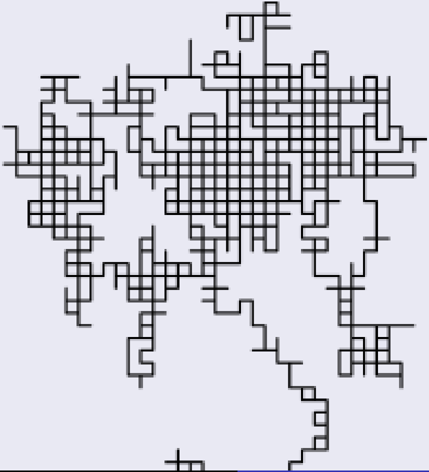
2-dim



Introduction(Continued)

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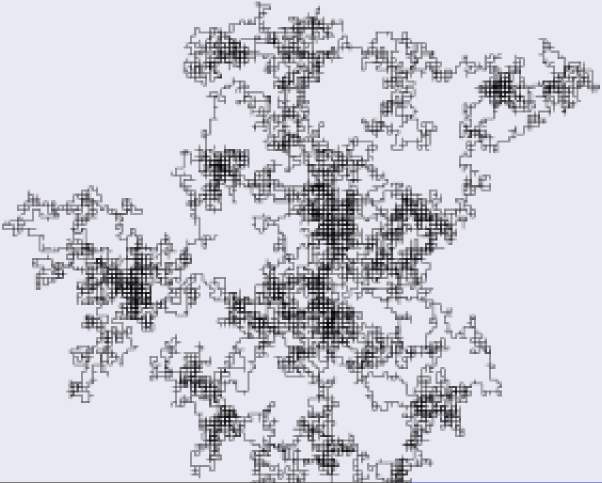
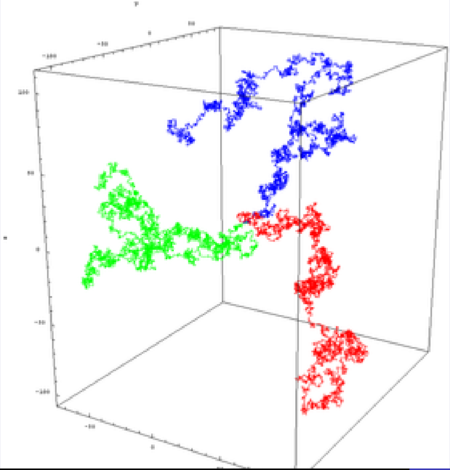


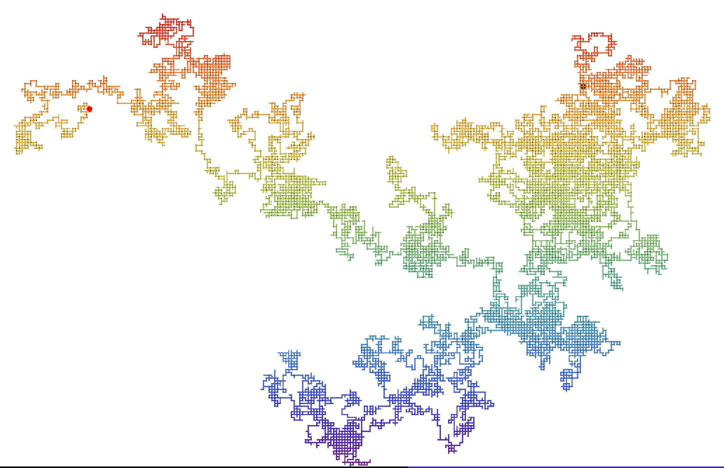
Introduction(Continued)

3-dim

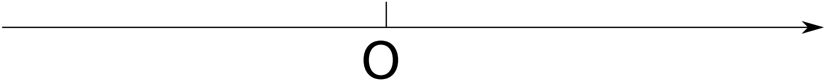


Example

π-30,000 Steps



"Fair Coin"



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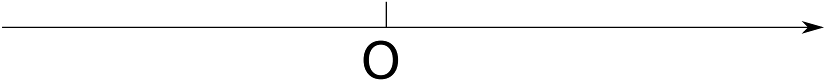

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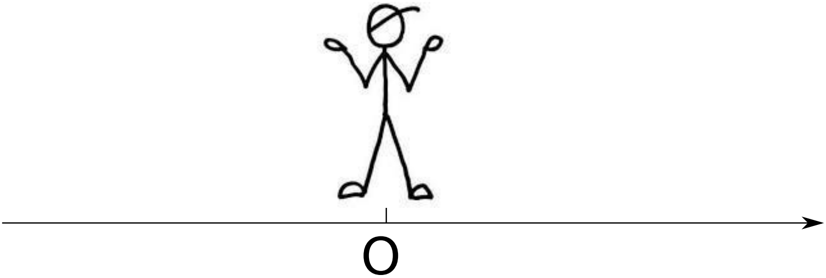

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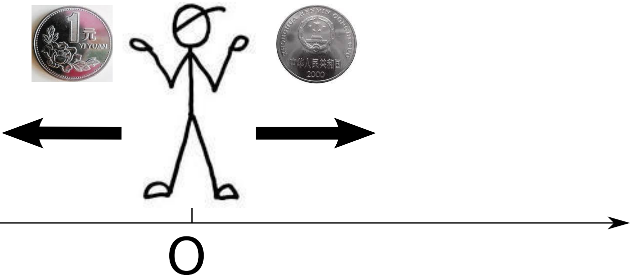
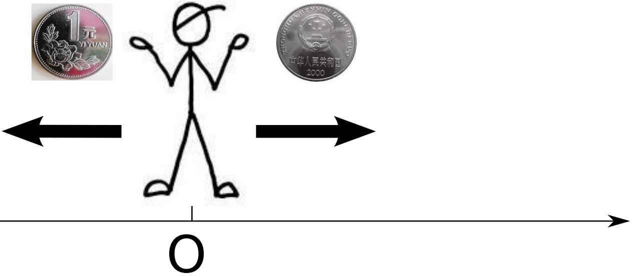


"Fair Coin"

"Fair": $P(H) = \frac{1}{2} = P(T)$.



Euler Polynomials

DEF.

Euler Polynomials $E_n(x)$:

$$\sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!} = \frac{2}{e^z+1} e^{xz} = \frac{e^{z\left(x-\frac{1}{2}\right)}}{\cosh\left(\frac{z}{2}\right)}.$$

Generalized Euler Polynomials: $E_n^{(p)}(x)$:

$$\sum_{n=0}^{\infty} E_n^{(p)}(x) \frac{z^n}{n!} = \left(\frac{2}{e^z+1}\right)^p e^{xz} = \frac{e^{z\left(x-\frac{p}{2}\right)}}{\cosh^p\left(\frac{z}{2}\right)}.$$

THM.

$$E_n^{(p)}(x) = \sum_{k_1+\cdots+k_p=n} \binom{n}{k_1,\ldots,k_p} E_{k_1}(x) E_{k_2}(0) \cdots E_{k_p}(0).$$

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Euler Polynomials

Question

Reciprocal Formula

$$E_n(x) = f\left(E_{n_1}^{(p_1)}(x), \ldots, E_{n_k}^{(p_k)}(x)\right)?$$

THM.[L. J., C. Vignat, V. H. Moll]

$$E_n(x) = \frac{1}{N^n} \sum_{n=N}^{\infty} p_l^{(N)} E_n^{(l)}\left(\frac{l-N}{2} + N \cdot x\right).$$

Here,

$$\frac{1}{T_N\left(\frac{1}{z}\right)} = \sum_{l=N}^{\infty} p_l^{(N)} z^l, \quad T_n(\cos\theta) = \cos(n\theta).$$

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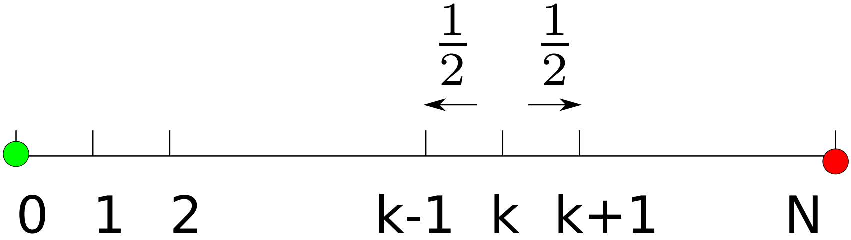

- 0 is the source and N is the sink;
- fair coin;
- ν_N: random number of steps for this process $\nu_N \geq N$.

Fact

$$p_l^{(N)} = P(\nu_N = l) = \frac{1}{N}\sum_{k=1}^{N}(-1)^{k+1}\sin\left(\theta_k^{(N)}\right)\cos^{l-1}\left(\theta_k^{(N)}\right), \ \theta_k^{(N)} = \frac{\pi}{2}\cdot\frac{2k-1}{N}.$$

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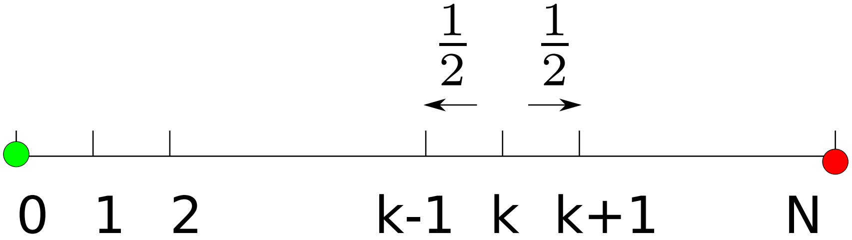

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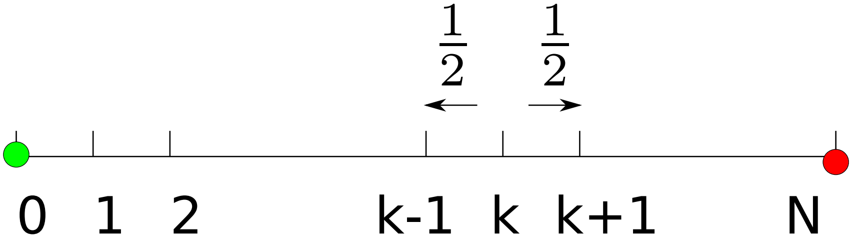

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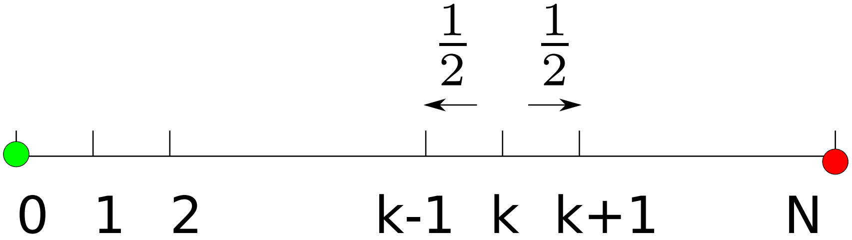

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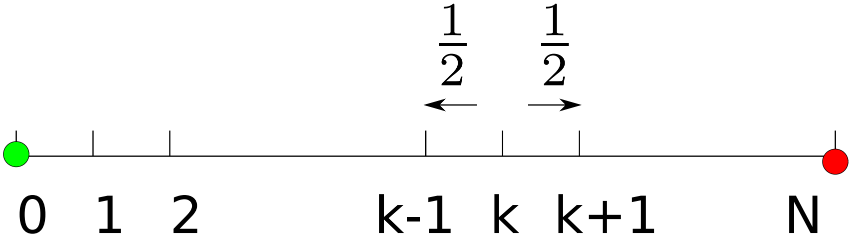

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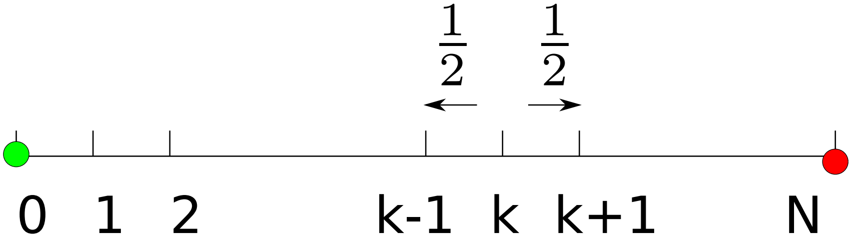

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Probabilistic Point of View

Consider random variable $L \sim p_L(x)$, for $p_L(x) = \operatorname{sech}(\pi x)$. Then,

1. $E_n(x) = \mathbb{E}\left[\left(x + \iota L - \frac{1}{2}\right)^n\right] = \int_{\mathbb{R}} \left(x + \iota L - \frac{1}{2}\right)^n \operatorname{sech}(\pi x)\, dx$;
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4. a Brownian motion interpretation exits.

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$$\mathcal{B} \sim \iota \tilde{L} - \frac{1}{2}, \ \tilde{L} \sim \tilde{p}(x) = \operatorname{sech}^2(\pi x).$$

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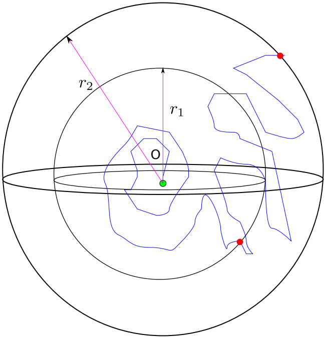

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- [A. N. Borodin and P. Salminen] Handbook of Brownian Motion-Facts and Formulae

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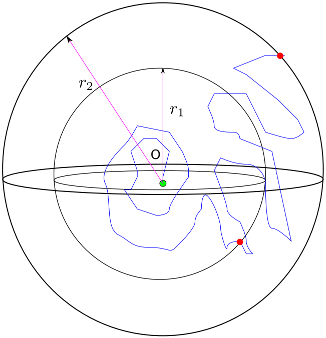

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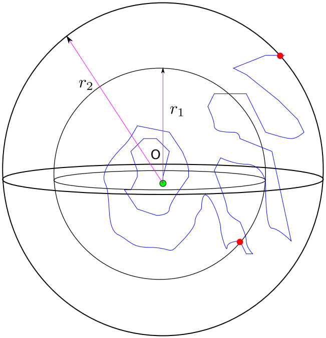

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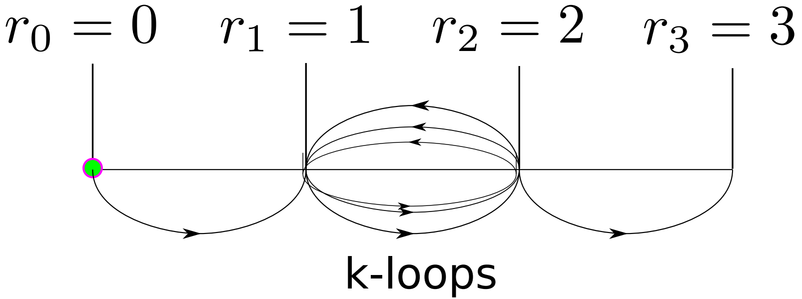

Formulae & Results

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THM.[L.J., C. Vignat]

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REMARK: Results for general $r_0, \ldots, r_3$ and also for arbitarily finite levels: $r_0, \ldots, r_N$ are also obtained.

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Conclusion

- Finding the proper RW is important, or namely, the results obained are important.
- RW on surfaces/manifolds?
- Information Geometry on hyperbolic secant (square) density?

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Thank You!