## Random Walk: A Probabilistic and Geometric Approach to Number Theory

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#### Acknowledgement

Introduction 1-dim (finite) RW & (Generalized) Euler Polynomials 3-dim RW and (Generalized) Bernoulli Polynomials Conclusion

### Acknowledgement

#### Joint Work with:



Christophe Vignat



Victor Hugo Moll

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cknowledgement Introduction

1-dim (finite) RW & (Generalized) Euler Polynomials 3-dim RW and (Generalized) Bernoulli Polynomials Conclusion Examples Simple "Fair Coin" Example

#### Introduction

#### 2-dim



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### Introduction(Continued)

#### 2-dim



Examples Simple "Fair Coin" Example

### Introduction(Continued)

#### 3-dim



Examples Simple "Fair Coin" Exampl

### Example

#### $\pi$ -30,000 Steps



Examples Simple "Fair Coin" Example

#### "Fair Coin"



www.alamy.com - DWISME



Examples Simple "Fair Coin" Example

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Defintion and Results Connection to Random Walk

#### **Euler Polynomials**

#### DEF.

Euler Polynomials  $E_n(x)$ :

$$\sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!} = \frac{2}{e^z + 1} e^{xz} = \frac{e^{z\left(x - \frac{1}{2}\right)}}{\cosh\left(\frac{z}{2}\right)}.$$

Generalized Euler Polynomials:  $E_n^{(p)}(x)$ :

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#### THM.

$$E_{n}^{(p)}(x) = \sum_{k_{1}+\dots+k_{p}=n} {n \choose k_{1},\dots,k_{p}} E_{k_{1}}(x) E_{k_{2}}(0) \cdots E_{k_{p}}(0).$$

Defintion and Results Connection to Random Walk

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#### Question

Reciprocal Formula

$$E_{n}(x) = f\left(E_{n_{1}}^{(p_{1})}(x), \ldots, E_{n_{k}}^{(p_{k})}(x)\right)?$$

#### THM.[L. J., C. Vignat, V. H. Moll]

$$E_n(x) = \frac{1}{N^n} \sum_{n=N}^{\infty} p_l^{(N)} E_n^{(l)} \left( \frac{l-N}{2} + N \cdot x \right).$$

$$\frac{1}{T_N\left(\frac{1}{z}\right)} = \sum_{l=N}^{\infty} p_l^{(N)} z^l, \ T_n\left(\cos\theta\right) = \cos\left(n\theta\right).$$

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0 is the source and N is the sink;

■ fair coin;

•  $\nu_N$ : random number of steps for this process  $\nu_N \ge N$ .

Defintion and Results Connection to Random Walk

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Fact  

$$p_{l}^{(N)} = P(\nu_{N} = l) = \frac{1}{N} \sum_{k=1}^{N} (-1)^{k+1} \sin\left(\theta_{k}^{(N)}\right) \cos^{l-1}\left(\theta_{k}^{(N)}\right), \ \theta_{k}^{(N)} = \frac{\pi}{2} \cdot \frac{2k-1}{N}.$$

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Definiton and Results Connection to Random Walk

### Probabilistic Point of View

Consider random variable  $L \sim p_L(x)$ , for  $p_L(x) = \operatorname{sech}(\pi x)$ . Then, **1**  $E_n(x) = \mathbb{E}\left[\left(x + \iota L - \frac{1}{2}\right)^n\right] = \int_{\mathbb{R}} \left(x + \iota L - \frac{1}{2}\right)^n \operatorname{sech}(\pi x) dx;$   $\mathcal{E} = \iota L - \frac{1}{2} \Rightarrow E_n(x) = \mathbb{E}\left[\left(\mathcal{E} + x\right)^n\right]$ **2**  $E^{(p)}(x) - \mathbb{E}\left[\left(x + (\iota L - \frac{1}{2}) + \dots + (\iota L - \frac{1}{2})\right)^n\right]$  for i.i.d  $\mathcal{E}(x)^p$ .

2  $E_n^{(p)}(x) = \mathbb{E}\left[\left(x + (\iota L_1 - \frac{1}{2}) + \dots + (\iota L_p - \frac{1}{2})\right)^n\right]$  for i.i.d  $\{L_i\}_{i=1}^p$ ; 3 Klebanov et al.<sup>1</sup> consider i.i.d.  $\{L_i\}_{i=1}^\infty$ :

$$\mathbb{E}(z^{\nu_N}) = \frac{1}{T_N\left(\frac{1}{z}\right)}, \ Z_N := \frac{1}{N} \sum_{n=1}^{\nu_N} L_n \sim L;$$

4 a Brownian motion interpretation exits.

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Connection to Random Walk

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Bernoulli Polynomials 3-dim RW/Brownian Motion

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Bernoulli Polynomials 3-dim RW/Brownian Motion

#### End



#### Hitting Time: the first time it "hits"

Denote:

 $\int H_z := \inf \{t : |X_t| = z\}$  Hitting Time  $S_z$ 

 $\mathbb{E}_{x}(H_{z}: \text{Requirement})$  Starting at  $|X_{t}| = x$ 

 [A. N. Borodin and P. Salminen]
 Handbook of Brownian Motion-Facts and Formulae

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3-dim RW/Brownian Motion

#### End



- Hitting Time: the first time it "hits" Denote:
  - $\begin{cases} H_z := \inf \{ t : |X_t| = z \} & \text{Hitting Time } S_z \\ \mathbb{E}_x (H_z : \text{Requirement}) & \text{Starting at } |X_t| = x \end{cases}$

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Bernoulli Polynomials 3-dim RW/Brownian Motion

### Different Angle



3-dim RW/Brownian Motion

#### Formulae & Results

$$\begin{cases} \mathbb{E}_{x} \left( e^{-\alpha H_{z}}; H_{z} < \infty \right) = \begin{cases} \frac{z \sinh(x\sqrt{2\alpha})}{x \sinh(z\sqrt{2\alpha})}, & 0 \le x \le z \\ \frac{z}{x} e^{-(x-z)\sqrt{2\alpha}}, & z \le x \end{cases} \\ \mathbb{E}_{x} \left( e^{-\alpha H_{z}}; \sup_{0 \le s \le H_{z}} R_{s}^{(3)} < y \right) = \begin{cases} \frac{z \sinh(x\sqrt{2\alpha})}{x \sinh(z\sqrt{2\alpha})}, & 0 \le x \le z \le y \\ \frac{z \sinh((y-x)\sqrt{2\alpha})}{x \sinh((y-z)\sqrt{2\alpha})}, & z \le x \le y \end{cases} \\ \mathbb{E}_{x} \left( e^{-\alpha H_{z}}; \inf_{0 \le s \le H_{z}} R_{s}^{(3)} > y \ge 0 \right) = \begin{cases} \frac{z \sinh((x-y)\sqrt{2\alpha})}{x \sinh((z-y)\sqrt{2\alpha})}, & y \le x \le z \\ \frac{z \sinh((x-y)\sqrt{2\alpha})}{x \sinh((z-y)\sqrt{2\alpha})}, & y \le x \le z \end{cases} \end{cases}$$

$$\frac{3^n}{n+1} \left[ B_{n+1} \left( \frac{x}{6} + \frac{5}{6} \right) - B_{n+1} \left( \frac{x}{6} + \frac{1}{2} \right) \right] = \frac{1}{4} \sum_{k \ge 0} \frac{1}{4^k} E_n^{(2k+2)} \left( \frac{x+2k+3}{2} \right) .$$

Bernoulli Polynomials 3-dim RW/Brownian Motion

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#### THM.[L.J., C. Vignat]

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REMARK: Results for general  $r_0, \ldots, r_3$  and also for arbitarily finite levels:  $r_0, \ldots, r_N$  are also obtained.

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- RW on surfaces/manifolds?
- Information Geometry on hyperbolic secant (square) density?



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# Thank You!

Lin Jiu Random Walk: A Probabilistic and Geometric Approach to Num

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