

# Binomial Identity in Arbitrary Bases

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## Binomial Identity

$$(X + Y)^n = \sum_{k=0}^n \binom{n}{k} X^k Y^{n-k}.$$

Generalization.

## Multi-nomial Identity

$$(X_1 + \cdots + X_m)^n = \sum_{k_1 + \cdots + k_m = n} \binom{n}{k_1, \dots, k_m} X_1^{k_1} \cdots X_m^{k_m}.$$

$n$	0	1	2	3	4	5	6	7	...
$(n)_2$	0	1	10	11	100	101	110	111	...

DEF.

$S_2(n) = \#$  of 1's in the binary expansion of  $n$ .

Example

$$\begin{cases} S_2(3) = 2 \\ S_2(4) = 1 \\ S_2(7) = 3 \end{cases}$$

## Theorem

$$(X + Y)^{S_2(n)} = \sum_{0 \leq k \leq n \text{ \& \& some condition}} X^{S_2(k)} Y^{S_2(n-k)},$$

some condition = there is no carry when adding  $k + (n - k) = n$ .

In fact,

$$k + (n - k) = n \text{ is carry free} \Leftrightarrow S_2(k) + S_2(n - k) = S_2(n).$$

Thus,

$$(X + Y)^{S_2(n)} = \sum_{S_2(k) + S_2(n-k) = S_2(n)} X^{S_2(k)} Y^{S_2(n-k)}.$$

# Example

## Example

$n = 6 = (110)_2$ . Therefore,

$$LHS = (X + Y)^{S_2(2)} = (X + Y)^2.$$

On the other hand,

$k + (n - k)$	0 + 6	1 + 5	2 + 4	3 + 3	4 + 2	5 + 1	6 + 0
	000	001	010	011	100	101	110
Binary	$\frac{110}{110}$	$\frac{101}{110}$	$\frac{100}{110}$	$\frac{011}{110}$	$\frac{010}{110}$	$\frac{001}{110}$	$\frac{000}{110}$
Carry-free	✓	x	✓	x	✓	x	✓

$$\begin{aligned} RHS &= X^{S_2(0)} Y^{S_2(6)} + X^{S_2(2)} Y^{S_2(4)} + X^{S_2(4)} Y^{S_2(2)} + X^{S_2(6)} Y^{S_2(0)} \\ &= Y^2 + XY + XY + X^2. \end{aligned}$$

## Remark

Carry Free  $\Leftrightarrow 1 + 1$  does not appear.

## Theorem

$$(X + Y)^{S_2(n)} = \sum_{S_2(k) + S_2(n-k) = S_2(n)} X^{S_2(k)} Y^{S_2(n-k)}.$$

carry-free  $\Leftrightarrow \binom{S_2(n)}{i}$ , i.e., distributing 1's in  $(n)_2$ .

- How to generalize this result to other bases rather than 2, for example 3?
- How to define

$$S_3(n) = ???$$

# of 1's? # of 2's? # of non-zero digits?

- In binary,  $0 + 1$ ,  $0 + 0$  are OK and  $1 + 1$  is not.  $\Leftrightarrow$  Carry Free.  
In base 3,

$$1 + 1 = 2?$$

$S_2(n) = \#$  of 1's in the binary expansion of  $n =$  sum of all digits.

Thus,

$S_b(n) =$  sum of all digits of  $n$  in its expansion of base  $b$ .

This also implies  $1 + 1 = 2$  is allowed.

Unfortunately  $(X + Y)^{S_3(n)} \neq \sum_{\text{carry-free}} X^{S_3(k)} Y^{S_3(n-k)}$

$S_3(2) = S_3((2)_3) = 2.$

$k + (n - k)$	$0 + 2$	$1 + 1$	$2 + 0$
	0	1	2
Binary	$\frac{2}{2}$	$\frac{1}{2}$	$\frac{0}{2}$
Carry-free	✓	✓	✓

$LHS = (X + Y)^2 \neq X^2 + XY + Y^2 = RHS.$

## Result

$$(X + Y)^{S_b(n)} = \sum_{\square_1} \square_2 X^{S_b(k)} Y^{S_b(n-k)}$$

$$\begin{cases} \square_1 = \text{carry-free} & \mapsto? \\ \square_2 = 1 & \mapsto? \end{cases}$$

Modify  $\square_2$ :

$$(X + Y)^{S_b(n)} = \sum_{\text{carry-free}} \left( \prod_{j=0}^{b-1} (j!)^{S_b^{(j)}(n) - S_b^{(j)}(k) - S_b^{(j)}(n-k)} \right) X^{S_b(k)} Y^{S_b(n-k)},$$

where

$$S_b^{(j)}(m) = \# \text{ of } j\text{'s in } m.$$



# Result

$$n = 2$$

$$S_3(2) = 2.$$

$k + (n - k)$	$0 + 2$	$1 + 1$	$2 + 0$
	0	1	2
Binary	$\frac{2}{2}$	$\frac{1}{2}$	$\frac{0}{2}$
Carry-free	✓	✓	✓

$$\begin{cases} LHS = (X + Y)^2 \\ RHS = X^2 + Y^2 + (2!)^{1-0-0} XY \end{cases}$$

## Next

Define the  $b$ -ary binomial coefficients:

$$\binom{n}{k}_b := \begin{cases} \prod_{j=0}^{b-1} (j!)^{S_b^{(j)}(n) - S_b^{(j)}(k) - S_b^{(j)}(n-k)} & \text{if carry-free} \\ 0 & \text{otherwise} \end{cases}$$

$$(X + Y)^{S_b(n)} = \sum_{k=0}^n \binom{n}{k}_b X^{S_b(k)} Y^{S_b(n-k)}.$$



## Generating Function

Define

$$f(n, b, x) := \sum_{k=0}^n \binom{n}{k}_b x^k,$$

for  $b = 4$ ,

$n$	$f(n, 4, x)$
1	$1 + x$
2	$(1 + x)^2$
3	$(1 + x)^3$
4	$1 + x^4$
5	$(1 + x)(1 + x^4)$
6	$(1 + x)^2(1 + x^4)$
7	$(1 + x)^3(1 + x^4)$
8	$(1 + x^4)^2$

## Theorem

$$(X + Y)^{S_b(n)} = \sum_{k=0}^n \binom{n}{k}_b X^{S_b(k)} Y^{S_b(n-k)},$$

where

$$\binom{n}{k}_b = \begin{cases} \prod_{j=0}^{b-1} (j!)^{S_b^{(j)}(n) - S_b^{(j)}(k) - S_b^{(j)}(n-k)} & \text{carry-free} \\ 0 & \text{otherwise} \end{cases} = \prod_{l=0}^{N-1} \binom{n_l}{k_l}.$$

In base  $b$ ,

$$n = n_{N-1}n_{N-2} \cdots n_0 \text{ and } k = k_{N-1}k_{N-2} \cdots k_0$$

i.e.,

$$n = n_{N-1}b^{N-1} + n_{N-2}b^{N-2} + \cdots + n_0b^0 \text{ and } k = \sum_{j=0}^{N-1} k_j b^j.$$

$$\text{carry-free} \Leftrightarrow k_i \leq n_i$$

## Theorem

$$(X + Y)^{S_b(n)} = \sum_{k=0}^n \binom{n}{k}_b X^{S_b(k)} Y^{S_b(n-k)}, \quad \binom{n}{k}_b = \prod_{l=0}^{N-1} \binom{n_l}{k_l}$$

Implies:

$$\sum_{k=0}^n \binom{n}{k}_b x^k = \prod_{l=0}^{N-1} (1 + x^{b^l})^{n_l}.$$

## Remark[Lucas Theorem]

For a prime  $p$ ,

$$\binom{n}{k}_p \equiv \prod_{l=0}^{N-1} \binom{n_l}{k_l} \pmod{p}.$$

Thank You!