"Random Walks" for Harmonic Sums

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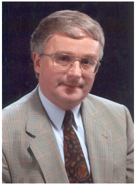
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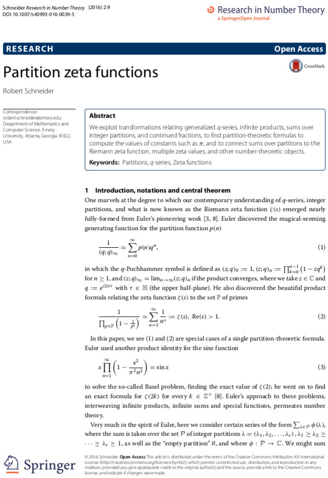
Outlines

Beginning-Partition



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Partition zeta functions

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Abstract

We exploit transformations relating generalized q-series, infinite products, sums over integer partitions, and continued fractions, to find partition-theoretic formulas to compute the values of constants such as π, and to connect sums over partitions to the Riemann zeta function, multiple zeta values, and other number-theoretic objects.

Keywords: Partitions, q-series, Zeta functions

1 Introduction, notations and central theorem

One marvels at the degree to which our contemporary understanding of q-series, integer partitions, and what is now known as the Riemann zeta function $\zeta(s)$ emerged nearly fully-formed from Euler's pioneering work [3, 8]. Euler discovered the magical-seeming generating function for the partition function $p(n)$

$$\frac{1}{(q;q)_\infty} = \sum_{n=0}^{\infty} p(n) q^n, \tag{1}$$

in which the q-Pochhammer symbol is defined as $(z;q)_0 := 1$, $(z;q)_n := \prod_{k=0}^{n-1}(1-zq^k)$ for $n \geq 1$, and $(z;q)_\infty = \lim_{n\to\infty}(z;q)_n$ if the product converges, where we take $z \in \mathbb{C}$ and $q := e^{i2\pi\tau}$ with $\tau \in \mathbb{H}$ (the upper half-plane). He also discovered the beautiful product formula relating the zeta function $\zeta(s)$ to the set $\mathbb{P}$ of primes

$$\frac{1}{\prod_{p\in\mathbb{P}}\left(1-\frac{1}{p^s}\right)} = \sum_{n=1}^{\infty}\frac{1}{n^s} := \zeta(s), \ \mathrm{Re}(s) > 1. \tag{2}$$

In this paper, we see (1) and (2) are special cases of a single partition-theoretic formula. Euler used another product identity for the sine function

$$x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{\pi^2 n^2}\right) = \sin x \tag{3}$$

to solve the so-called Basel problem, finding the exact value of $\zeta(2)$; he went on to find an exact formula for $\zeta(2k)$ for every $k \in \mathbb{Z}^+$ [8]. Euler's approach to these problems, interweaving infinite products, infinite sums and special functions, permeates number theory.

Very much in the spirit of Euler, here we consider certain series of the form $\sum_{\lambda\in\mathcal{P}}\phi(\lambda)$, where the sum is taken over the set $\mathcal{P}$ of integer partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$, as well as the "empty partition" $\emptyset$, and where $\phi : \mathcal{P} \to \mathbb{C}$. We might sum

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- R. Schneider, Partition zeta functions. *Research in Number Theory* 2016, 2:8.
- Let $\varphi_\infty(f;q) = \prod_{n=1}^{\infty}(1-f(n)q^n)$:

$$\frac{1}{\varphi_\infty(f;q)} = \sum_{\lambda\in\mathcal{P}} q^{|\lambda|}\prod_{\lambda_i\vdash\lambda} f(\lambda_i).$$

- For $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$, we denote

$$|\lambda| = n, \ l(\lambda) := k \text{ and } n_\lambda := \lambda_1\cdots\lambda_k$$

Define the *partition-theoretic generaliztion of Riemann-zeta function* as

$$\zeta_{\mathcal{P}}\left(\{a\}^k\right) := \sum_{l(\lambda)=k}\frac{1}{n_\lambda^a}.$$

Harmonic Sums

DEF: harmonic sum

$$S_{a_1,\ldots,a_k}(N)=\sum_{N\geq n_1\geq\cdots\geq n_k\geq 1}\frac{\operatorname{sign}(a_1)^{n_1}}{n_1^{|a_1|}}\cdots\frac{\operatorname{sign}(a_k)^{n_k}}{n_k^{|a_k|}}.$$

$k=1,\ a_1>0,\ N=\infty$

$$S_{a_1}(\infty)=\sum_{n_1=1}^{\infty}\frac{1}{n_1^{a_1}}=\zeta(a_1).$$

$a_1=\cdots=a_k=a>0,\ N=\infty$

$$S_{a_k}(\infty)=S_{\underbrace{a,\ldots,a}_{k}}(\infty)=\sum_{n_1\geq\cdots\geq n_k\geq 1}\frac{1}{(n_1\cdots n_k)^a}.$$

Recall

$$\zeta_{\mathcal{P}}\left(\{a\}^k\right):=\sum_{l(\lambda)=k}\frac{1}{n_\lambda^a}=\sum_{\lambda_1\geq\cdots\geq\lambda_k\geq 1}\frac{1}{(\lambda_1\cdots\lambda_k)^a}.$$

Generating Function

Fact

$$\zeta_{\mathcal{P}}\left(\{a\}^k\right)=S_{a_k}(\infty)$$

$$\prod_{n=1}^{\infty}\frac{1}{1-\frac{t^a}{n^a}}=\sum_{k=0}^{\infty}\sum_{l(\lambda)=k}\frac{t^{ak}}{n_\lambda^a}=\sum_{k=0}^{\infty}S_{a_k}(\infty)\,t^{ak}.$$

In particular, if $a=m\in\mathbb{N}$ and $m\geq 2$, by considering $\xi_m:=\exp\left(\frac{2\pi\iota}{m}\right)$ (M. Chamberland and A. Straub)

$$\sum_{k=0}^{\infty}S_{m_k}(\infty)\,t^{mk}=\prod_{n=1}^{\infty}\frac{n^m}{n^m-t^m}=\prod_{n=1}^{\infty}\frac{n^m}{\left(n-\xi_m^0 t\right)\cdots\left(n-\xi_m^{m-1}t\right)}=\prod_{j=0}^{m-1}\Gamma\left(1-\xi_m^j t\right).$$

Integral Representation

Blümlein wrote me a hand writing notes on

$$B(N,1+t)=\frac{1}{N}\sum_{k=0}^{\infty}(-t)^k S_{\mathbf{1}_k}(N).$$

$m=2$

$$\begin{aligned}\sum_{k=0}^{\infty}S_{\mathbf{2}_k}(\infty)t^{2k} &= \Gamma(1+t)\Gamma(1-t)=B(1+t,1-t)\\ &= \int_0^1 \lambda^{-t}(1-\lambda)^t\,d\lambda=\sum_{k=0}^{\infty}\frac{t^k}{k!}\int_0^1 \ln^k\left(\frac{1-\lambda}{\lambda}\right)d\lambda.\end{aligned}$$

$$S_{\mathbf{2}_k}(\infty)=\frac{1}{(2k)!}\int_0^1 \ln^{2k}\left(\frac{1-\lambda}{\lambda}\right)d\lambda.$$

In particular, $k=1$ yields, for Riemann-zeta function ζ:

$$\frac{\pi^2}{6}=\zeta(2)=S_2(\infty)=\frac{1}{2}\int_0^1 \ln^2\left(\frac{1-\lambda}{\lambda}\right)d\lambda.$$

Integral Representation

Multiple Beta Function

$$B(\alpha_1, \ldots, \alpha_m) := \frac{\Gamma(\alpha_1) \cdots \Gamma(\alpha_m)}{\Gamma(\alpha_1 + \cdots + \alpha_m)} = \int_{\Omega_m} \prod_{i=1}^{m} x_i^{\alpha_i - 1} d\boldsymbol{x},$$

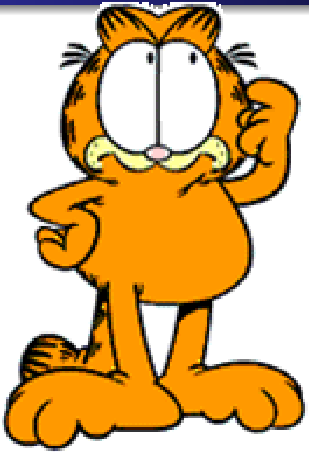
where $\Omega_m = \left\{(x_1, \ldots, x_m) \in \mathbb{R}_+^m : x_1 + \cdots + x_{m-1} < 1,\ x_1 + \cdots + x_m = 1\right\}$.

Prop.

$$S_{\boldsymbol{m}_k}(\infty) = \frac{(-1)^{mk}(m-1)!}{(mk)!} \int_{\Omega_m} \ln^{mk}\left(\prod_{j=0}^{m-1} x_{j+1}^{\xi_m^j}\right) d\boldsymbol{x},\ \ \xi_m = \exp\left(\frac{2\pi\iota}{m}\right)$$

$$\begin{aligned}\zeta(m) &= \frac{(-1)^m}{m} \int_0^1 \int_0^{1-x_1} \cdots \int_0^{1-x_1-\cdots-x_{m-2}} \\ &\quad \ln^m\left(x_1^{\xi_m^0} \cdots x_{m-1}^{\xi_m^{m-2}} (1 - x_1 - \cdots - x_{m-1})^{\xi_m^{m-1}}\right) dx_{m-1} \cdots dx_1.\end{aligned}$$

BREAK



$S_{1_k}(N) = \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1 \cdots n_k}$

Label N sites as follows:

$$\begin{array}{cccccc} \bullet & \bullet & \bullet & \cdots & \bullet & \bullet \\ 1 & 2 & 3 & & N-1 & N \end{array}$$

We start a random walk at site "N", with the rules: (as a pawn)

$$\mathbb{P}(i \to j) = \text{the probability from site "}i\text{" to site "}j\text{"} = \begin{cases} 1/i, & \text{if } j \leq i; \\ 0, & \text{if } j > i. \end{cases}$$

namely:

- one can only jump to sites that are NOT to the right of the current site, with equal probabilities;
- steps are independent.

For example, suppose we are at site "6":

$$\begin{array}{cccccccccc} & & & & & \text{here} & & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & & N \end{array}$$

Then, the next step only allows to walk to sites $\{1, 2, 3, 4, 5, 6\}$, with probabilities:

$$\mathbb{P}(6 \to 6) = \mathbb{P}(6 \to 5) = \mathbb{P}(6 \to 4) = \mathbb{P}(6 \to 3) = \mathbb{P}(6 \to 2) = \mathbb{P}(6 \to 1) = \frac{1}{6}.$$

$S_{1_k}(N) = \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1 \cdots n_k}$

$$\begin{array}{cccccc} \bullet & \bullet & \bullet & \cdots & \bullet & \bullet \\ 1 & 2 & 3 & & N-1 & N \end{array}$$

Therefore, a typical walk is as follows:

STEP 1: walk from site "N" to some site "$n_1 (\leq N)$", with $\mathbb{P}(N \to n_1) = \frac{1}{N}$;

STEP 2: walk from site "n_1" to some site "$n_2 (\leq n_1)$", with $\mathbb{P}(n_1 \to n_2) = \frac{1}{n_1}$;

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

STEP $k+1$: walk from site "n_k" to site "$n_{k+1} (\leq n_k)$", with $\mathbb{P}(n_k \to n_{k+1}) = \frac{1}{n_k}$.

Focus on $\mathbb{P}(n_{k+1} = 1)$:

$$\mathbb{P}(n_{k+1} = 1) = \sum_{\text{All possible paths}} \frac{1}{N} \frac{1}{n_1} \cdots \frac{1}{n_k} = \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{N n_1 \cdots n_k} = \frac{S_{1_k}(N)}{N}.$$

$S_{\mathbf{1}_k}(N)$

On the other hand, the transition matrix of sites $\{1,\ldots,N\}$ is:

$$P_{N|1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \end{pmatrix}$$

i.e.,

$$P_{N|1} = \left(p_{i,j}^{(1)}\right) \text{ with } p_{ij}^{(1)} = \mathbb{P}\left(i \to j\right) = \frac{1}{i}.$$

Therefore, after $k+1$ steps, entries of $P_{N|1}^{k+1}$ give the transition probabilities among sites. In particular,

$$\left(P_{N|1}^{k+1}\right)_{N,1} = \mathbb{P}\left(n_{k+1} = 1\right) = \frac{1}{N} S_{\mathbf{1}_k}\left(N\right),$$

Matrix Representation

$$S_{\mathbf{1}_k}(N) \to S_{\mathbf{m}_k}(N) \to S_{\mathbf{a}_k}(N) \to S_{a_1,\ldots,a_k}(N)$$

Recall

$$S_{a_1,\ldots,a_k}(N) = \sum_{N\geq n_1\geq\cdots\geq n_k\geq 1} \frac{\operatorname{sign}(a_1)^{n_1}}{n_1^{|a_1|}} \times \cdots \times \frac{\operatorname{sign}(a_k)^{n_k}}{n_k^{|a_k|}}$$

For $l = 1, \ldots, k$

$$P_{N|a_l} = \begin{pmatrix} \operatorname{sign}(a_l) & 0 & \cdots & 0 \\ \frac{1}{2^{|a_l|}} & \frac{1}{2^{|a_l|}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\operatorname{sign}(a_l)^N}{N^{|a_l|}} & \frac{\operatorname{sign}(a_l)^N}{N^{|a_l|}} & \cdots & \frac{\operatorname{sign}(a_l)^N}{N^{|a_l|}} \end{pmatrix}.$$

THM.

Denote $a_0 = 1$, then

$$S_{a_1,\ldots,a_k}(N) = N \cdot \left(P_{N|a_0} P_{N|a_1} \cdots P_{N|a_k}\right)_{N,1} = N \cdot \left(\prod_{l=0}^{k} P_{N|a_l}\right)_{N,1}.$$

$S_{a_k}(N)$ with $a > 1$

$$M_{(N+1)|a} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ \frac{1}{2^a} & \frac{1}{2^a} & \cdots & 0 & 1-\frac{1}{2^{a-1}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{N^a} & \frac{1}{N^a} & \cdots & \frac{1}{N^a} & 1-\frac{1}{N^{a-1}} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = \left(\begin{array}{c|c} P_{N|a} & \overrightarrow{\left(1-\frac{1}{I^{a-1}}\right)} \\ \hline \underbrace{(0,\ldots,0)}_{N} & 1 \end{array}\right)$$

$$\bullet\ \bullet\ \bullet \cdots \bullet\ \bullet$$
$$1\ \ 2\ \ 3\ \ \ \ N\ \ \mathfrak{N}$$

with

$$\mathbb{P}(\mathfrak{N} \to \mathfrak{N}) = 1 \text{ and } \mathbb{P}(i \to \mathfrak{N}) = 1 - \frac{1}{i^{a-1}}.$$

$$\left(P_{N|a}^{k+1}\right)_{N,1} = \left(M_{(N+1)|a}^{k+1}\right)_{N,1} = \mathbb{P}(n_{k+1} = 1) = \frac{1}{N^a} S_{a_k}(N).$$

BREAK

Diagonalization

$a = 1$

$$P_{N|1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \end{pmatrix} \text{ and } \left(P_{N|1}^{k+1}\right)_{N,1} = \frac{1}{N} S_{\mathbf{1}_k}(N)$$

$$P_{N|1} = Q_{N|1} \operatorname{diag}\left\{1, \ldots, \frac{1}{N}\right\} Q_{N|1}^{-1} \Rightarrow P_{N|1}^{k+1} = Q_{N|1} \operatorname{diag}\left\{1, \ldots, \frac{1}{N^{k+1}}\right\} Q_{N|1}^{-1}.$$

$$Q_{N|1} = \left(\frac{\binom{i-1}{j-1}}{\binom{N-1}{j-1}}\right) \text{ and } Q_{N|1}^{-1} = \left((-1)^{i+j}\binom{N-1}{i-1}\binom{i-1}{j-1}\right)$$

$$\frac{1}{N} S_{\mathbf{1}_k}(N) = \left(P_{N|1}^{k+1}\right)_{N,1} = \sum_{l=1}^{N} \frac{1}{l^{k+1}} (-1)^{l-1} \binom{N-1}{l-1},$$

K. Dilcher:

$$\sum_{l=1}^{N} (-1)^{l-1} \binom{N}{l} \frac{1}{l^k} = \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1 \cdots n_k} = S_{\mathbf{1}_k}(N).$$

Diagonalization

$a > 0$

$$P_{N|a} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \frac{1}{2^a} & \frac{1}{2^a} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N^a} & \frac{1}{N^a} & \cdots & \frac{1}{N^a} \end{pmatrix} \text{ and } \left(P_{N|a}^{k+1}\right)_{N,a} = \frac{1}{N^a} S_{\mathbf{a}_k}(N).$$

Diagonalization implies:

$$S_{\mathbf{a}_k}(N) = \sum_{l=1}^{N} \left(\prod_{\substack{n=1 \\ n \neq l}}^{N} \frac{n^a}{n^a - l^a} \right) \frac{1}{l^{ak}}.$$

Recall

$$S_{\mathbf{1}_k}(N) = \sum_{l=1}^{N} (-1)^{l-1} \binom{N}{l} \frac{1}{l^k}.$$

When $a = 1$,

$$\prod_{\substack{n=1 \\ n \neq l}}^{N} \frac{n}{n-l} = (-1)^{l-1} \binom{N}{l}$$

More

$$\mathcal{S}(f;k;N) := \sum_{N\geq n_1\geq\cdots\geq n_k\geq 1} f_1(n_1)\cdots f_k(n_k).$$

Define $f_0(x) = \frac{1}{x}$ and for $l = 0, \ldots, k$

$$\mathcal{P}_{N|f_l} := \begin{pmatrix} f_l(1) & 0 & \cdots & 0 \\ f_l(2) & f_l(2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f_l(N) & f_l(N) & \cdots & f_l(N) \end{pmatrix}.$$

THM.

1. It holds that

$$\mathcal{S}(f;k;N) = N\cdot\left(\prod_{l=0}^{k}\mathcal{P}_{N|f_l}\right)_{N,1}.$$

2. If $\{f_l(1), \ldots, f_l(N)\}$ are all distinct, then

$$\mathcal{P}_{N,f_l} = \mathcal{Q}_{N,f_l}\operatorname{diag}\{f_l(1), \ldots, f_l(N)\}\,\mathcal{Q}_{N,f_l}^{-1}.$$

Entries are calculated explicitly.

$\mathcal{S}(f;k;N) := \sum_{N\geq n_1\geq\cdots\geq n_k\geq 1} f_1(n_1)\cdots f_k(n_k).$

- $k=1$ and $f_1(x)=x$, i.e.

$$\sum_{N\geq n_1\geq 1} n_1 = \frac{N(N+1)}{2} \Rightarrow \sum_{l=1}^{N}(-1)^{N-l}\, l^{N+1}\binom{N}{l} = \frac{N(N+1)!}{2};$$

- $f_1\equiv\cdots\equiv f_k = f$ and $f(m)=a_m$, $(a_m)_{m=1}^{N}$ of distinct numbers:

$$\sum_{N\geq n_1\geq\cdots\geq n_k\geq 1} a_{n_1}\cdots a_{n_k} = \sum_{j=1}^{N}\left(\prod_{\substack{m=1\\ m\neq j}}^{N}\frac{1}{1-\frac{a_m}{a_j}}\right)a_j^k,$$

a general result by Zeng: which, when taking $a_j=\frac{a-bq^{j+i-1}}{c-zq^{j+i-1}}$ and $N=n-i+1$, "turns out to be a common source of several q-identities"

What's Next?

- Is there a systematic "algorithm" to use this "diagonalization technique"?
- The integral representation leads to

$$B_{2k} = \frac{(-1)^{k+1}}{\left(1-2^{1-2k}\right)(2\pi)^{2k}} \int_0^1 \ln^{2k}\left(\frac{\lambda}{1-\lambda}\right) d\lambda.$$

- It holds that

$$\lim_{k\to\infty} S_{\mathbf{2}_k}(\infty) = 2 \Rightarrow \lim_{k\to\infty} \frac{1}{(2k)!} \int_0^1 \ln^{2k}\left(\frac{\lambda}{1-\lambda}\right) d\lambda = 2.$$

End

Thank You!