The Method of Brackets (MoB) and Integrating by Differentiating (IbD) Method

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Outlines

1. The method of brackets (MoB)
   - Rules
   - Ramanujan's Master Theorem (RMT)
   - Examples
   - Recent result
2. Integration by Differentiating
   - Formulas
   - Recent proofs
   - Connection

Rules

Idea

MoB evaluates the definite integral

$$\int_0^\infty f(x)\,dx$$

(most of the time) in terms of SERIES, with *ONLY SIX* rules:

Defintion [Indicator]

$$\phi_n := \frac{(-1)^n}{n!} = \frac{(-1)^n}{\Gamma(n+1)}$$

and

$$\phi_{1,\dots,r} := \phi_{n_1,\dots,n_r} = \phi_{n_1}\phi_{n_2}\cdots\phi_{n_r} = \prod_{i=1}^{r}\phi_{n_i}.$$

Rules (P-Production; E-Evaluation) $I=\int_0^\infty f(x)\,dx$

P_1: $f(x)=\sum_{n=0}^{\infty} a_n x^{\alpha n+\beta-1} \Rightarrow \int_0^\infty f(x)\,dx \mapsto \sum_n a_n \langle \alpha n+\beta\rangle$—Bracket Series;

P_2: For $\alpha<0$, $(a_1+\cdots+a_r)^\alpha \mapsto \sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} a_1^{n_1}\cdots a_r^{n_r} \frac{\langle -\alpha+n_1+\cdots+n_r\rangle}{\Gamma(-\alpha)}$;

P_3: For each bracket series, we assign *index=# of sums− # of brackets*;

E_1: $\sum_n \phi_n f(n)\langle \alpha n+\beta\rangle = \frac{1}{|\alpha|} f(n^*)\Gamma(-n^*)$, where n^* solves $\alpha n+\beta=0$;

E_2: $$\sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} f(n_1,\ldots,n_r) \prod_{i=1}^{r} \langle a_{i1}n_1+\cdots+a_{ir}n_r+c_i\rangle = \frac{f(n_1^*,\ldots,n_r^*)\prod_{i=1}^{r}\Gamma(-n_i^*)}{|\det A|},$$

$$(n_1^*,\ldots,n_r^*) \text{ solves } \begin{cases} a_{11}n_1+\cdots+a_{1r}n_r+c_1 & =0 \\ \ldots & \cdots \\ a_{r1}n_1+\cdots+a_{rr}n_r+c_r & =0 \end{cases};$$

E_3: The value of a multi-dimensional bracket series of ***POSITIVE*** index is obtained by computing all the contributions of maximal rank by Rule E_2. These contributions to the integral appear as series in the free parameters. Series converging in a common region are added and divergent series are discarded. Any series producing a non-real contribution is also discarded.

Ramanujan's Master Theorem[RMT]

Theorem[RMT]

$$\int_0^\infty x^{s-1}\left\{a(0)-\frac{a(1)}{1!}x+\frac{a(2)}{2!}x^2-\cdots\right\}dx=a(-s)\Gamma(s)$$

(1)

$$\int_0^\infty x^{s-1}\left(\sum_{n=0}^\infty \phi_n a(n)x^n\right)dx=a(-s)\Gamma(s)$$

(2) [Hardy]

•$H(\delta):=\{s=\sigma+\iota t:\sigma\geq-\delta,0<\delta<1\}$;

•$\psi(x)\in C^\infty(H(\delta))$; $\exists C,P,A,\ A<\pi$ such that $|\psi(s)|\leq Ce^{P\delta+A|t|}$, $\forall s\in H(\delta)$;

•$0<c<\delta,\ \Psi(x):=\frac{1}{2\pi\iota}\int_{c-\iota\infty}^{c+\iota\infty}\frac{\pi}{\sin(\pi s)}\psi(-s)x^{-s}dx \overset{0<x<e^{-P}}{=\!=\!=\!=} \sum_{k=0}^\infty\psi(k)(-x)^k$;

$$\int_0^\infty \Psi(x)x^{s-1}dx=\frac{\pi}{\sin(\pi s)}\psi(-s).$$

Rules Again

Theorem[RMT]

$$\int_0^\infty x^{s-1}\left\{\sum_{n=0}^\infty \phi_n a(n) x^n\right\} dx = a(-s)\Gamma(s)$$

(1) Integrand$\mapsto$Power Series;
(2) Keep Track of s;
(3) Apply the Formula;
(4) Multiple Integrals;

$$\int_0^\infty \int_0^\infty \sum_{n,m} a(m,n) x^m y^n dxdy = ?$$

(5) More Sums than Integrals (brackets);

$$\int_0^\infty f_1(x) f_2(x) dx = \int_0^\infty \sum_{m,n} a(m,n) x^{m+n} dx = \sum_{m,n} a(m,n) \langle m+n+1 \rangle = ?$$

(6) Extra.

Rules Again

P_1: $f(x)=\sum_{n=0}^{\infty} a_n x^{\alpha n+\beta-1} \Rightarrow \int_0^{\infty} f(x)\,dx \mapsto \sum_n a_n \langle \alpha n+\beta \rangle$ $\boxed{s-1\mapsto s}$

P_2: For $\alpha<0$, $(a_1+\cdots+a_r)^{\alpha} \mapsto \sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} a_1^{n_1}\cdots a_r^{n_r} \frac{\langle -\alpha+n_1+\cdots+n_r\rangle}{\Gamma(-\alpha)}$;

P_3: Index=# of sums− # of brackets; $\boxed{\text{Just a definition}}$

E_1: $\sum_n \phi_n f(n) \langle \alpha n+\beta\rangle = \frac{f(n^*)\Gamma(-n^*)}{|\alpha|}$,, n^* solves $\alpha n+\beta=0$; $\boxed{\text{RMT}}$

E_2: $\boxed{\text{Iteration of RMT}}$

$$\sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} f(n_1,\ldots,n_r) \prod_{i=1}^{r} \langle a_{i1}n_1+\cdots+a_{ir}n_r+c_i\rangle = \frac{f(n_1^*,\ldots,n_r^*)\prod_{i=1}^{r}\Gamma(-n_i^*)}{|\det A|}$$

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P_1: $f(x) = \sum_{n=0}^{\infty} a_n x^{\alpha n+\beta-1} \Rightarrow \int_0^\infty f(x)\,dx \mapsto \sum_n a_n \langle \alpha n+\beta \rangle$ $\boxed{s-1 \mapsto s}$

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Rule P_2

$$
\begin{aligned}
&\frac{\Gamma(-\alpha)}{(a_1+\cdots+a_r)^{-\alpha}}.\\
=\ &\int_0^\infty x^{-\alpha-1}e^{-(a_1+\cdots+a_r)x}dx\\
=\ &\int_0^\infty x^{-\alpha-1}e^{-a_1x}e^{-a_2x}\cdots e^{-a_rx}dx\\
=\ &\int_0^\infty x^{-\alpha-1}\prod_{i=1}^{r}\left(\sum_{n_i=0}^{\infty}\phi_{n_i}(ax)^{n_i}\right)dx\\
=\ &\int_0^\infty \sum_{n_1,\ldots,n_r}\phi_{1,\ldots,r}a_1^{n_1}\cdots a_r^{n_r}x^{n_1+\cdots+n_r-\alpha-1}dx\\
=\ &\sum_{n_1,\ldots,n_r}\phi_{1,\ldots,r}a_1^{n_1}\cdots a_r^{n_r}\langle-\alpha+n_1+\cdots+n_r\rangle
\end{aligned}
$$

Examples

$$I := \int_0^\infty xJ_0(xy)\frac{dx}{\sqrt{a^2+x^2}} = y^{-1}e^{-ay} \quad [y>0 \ Re(a)>0]$$

Rule P_2:

$$\frac{1}{\sqrt{a^2+x^2}} = \left(a^2+x^2\right)^{-\frac{1}{2}} = \sum_{n_1,n_2} \phi_{1,2} a^{2n_1} x^{2n_2} \frac{\left\langle \frac{1}{2}+n_1+n_2\right\rangle}{\Gamma\left(\frac{1}{2}\right)}$$

$J_0(xy)$

$$J_0(xy) = \sum_{n_3} \phi_{n_3} \frac{y^{2n_3}}{\Gamma(n_3+1)\, 2^{2n_3}} x^{2n_3}$$

Rule P_1

$$I = \int_0^\infty \sum_{n_1,n_2,n_3} \phi_{1,2,3} \frac{y^{2n_3} a^{2n_1}}{\Gamma(n_3+1)\Gamma\left(\frac{1}{2}\right) 2^{2n_3}} \left\langle n_1+n_2+\frac{1}{2}\right\rangle x^{2n_2+2n_3+1} dx$$

$$= \sum \phi_{1,2,3} \frac{y^{2n_3} a^{2n_1}}{\Gamma(n_3+1)\Gamma\left(\frac{1}{2}\right) 2^{2n_3}} \left\langle n_1+n_2+\frac{1}{2}\right\rangle \langle 2n_2+2n_3+2\rangle$$

Examples

$$I := \int_0^\infty x J_0(xy) \frac{dx}{\sqrt{a^2+x^2}} = y^{-1} e^{-ay}$$

$$I = \sum_{n_1,n_2,n_3} \phi_{1,2,3} \frac{y^{2n_3} a^{2n_1}}{\Gamma(n_3+1)\Gamma\left(\frac{1}{2}\right) 2^{2n_3}} \left\langle n_1+n_2+\frac{1}{2}\right\rangle \langle 2n_2+2n_3+2\rangle;$$

n_1 free: $n_2^* = -\frac{1}{2} - n_1$; $n_3^* = -\frac{1}{2} + n_1$; det $= 2$:

$$\begin{aligned} I &= \frac{1}{2}\sum_{n_1} \phi_{n_1} \frac{y^{2n_1-1} a^{2n_1}}{\Gamma\left(n_1+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) 2^{2n_1-1}} \Gamma\left(n_1+\frac{1}{2}\right)\Gamma\left(-n_1+\frac{1}{2}\right) \\ &= \frac{1}{y}\sum_{n_1=0}^{\infty} \phi_{n_1} \left(\frac{ay}{2}\right)^{2n_1} \frac{\Gamma\left(\frac{1}{2}-n_1\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{1}{y}\cosh(ay); \end{aligned}$$

n_2 free $: I = \frac{1}{\sqrt{\pi} y} \sum_{n_2=0}^{\infty} \frac{\Gamma\left(n_2+\frac{1}{2}\right)}{\Gamma(-n_2)} \left(\frac{2}{ay}\right)^{2n_2+1} = 0;$ n_3 free $: I = \text{Series} = -\frac{\sinh(ay)}{y};$

E_3 :

$$I = \frac{1}{y}\cosh(ay) - \frac{\sinh(ay)}{y} = y^{-1} e^{-ay}.$$

Multi-dim

$$I=\int_{\mathbb{R}^m} f\left(x_1^2+\cdots+x_m^2\right)dx_1\cdots dx_m$$

(I) Usual method: By the n-dim spherical coordinate that $r=x_1^2+\cdots+x_m^2$ and

$$\begin{cases} x_1=r\cos\left(\phi_1\right), & 0\leq\phi_1\leq\pi,\\ x_2=r\sin\left(\phi_2\right)\cos\left(\phi_2\right), & 0\leq\phi_2\leq\pi,\\ \cdots & \cdots\\ x_{n-2}=r\sin\left(\phi_1\right)\cdots\sin\left(\phi_{m-3}\right)\cos\left(\phi_{m-2}\right), & 0\leq\phi_{m-2}\leq\pi,\\ x_{n-1}=r\sin\left(\phi_1\right)\cdots\sin\left(\phi_{m-2}\right)\cos\left(\phi_{m-1}\right), & 0\leq\phi_{m-1}\leq 2\pi,\\ x_{n-1}=r\sin\left(\phi_1\right)\cdots\sin\left(\phi_{m-2}\right)\sin\left(\phi_{m-1}\right), & 0\leq r<\infty, \end{cases}$$

we have

$$dx_1\cdots dx_m=r^{m-1}\sin^{m-2}\left(\phi_1\right)\cdots\sin\left(\phi_{m-2}\right)drd\phi_1\cdots d\phi_{m-1}.$$

Thus,

$$I = 2\pi^{\frac{m}{2}}\left[\int_0^\infty r^{m-1}f\left(r^2\right)dr\right]\frac{1}{\Gamma\left(\frac{m}{2}\right)}.$$

Multi-dim

$$I=\int_{\mathbb{R}^m} f\left(x_1^2+\cdots+x_m^2\right)dx_1\cdots dx_m$$

(II) The method of brackets: Suppose

$$f(t)=\sum_{l=0}^{\infty}\phi_l a(l)\, t^l,$$

then,

$$\int_0^{\infty} r^{m-1} f\left(r^2\right)dr=\sum_l \phi_l a(l)\langle 2l+m\rangle=\sum_l \phi_l a(l)\langle 2l+m\rangle=\frac{1}{2}a\left(-\frac{m}{2}\right)\Gamma\left(\frac{m}{2}\right).$$

So it suffices to show that

$$I=2\pi^{\frac{m}{2}}\left[\frac{1}{2}a\left(-\frac{m}{2}\right)\frac{\Gamma\left(-\frac{m}{2}+1\right)}{(-1)^{-\frac{m}{2}}}\Gamma\left(\frac{m}{2}\right)\right]\frac{1}{\Gamma\left(\frac{m}{2}\right)}=\pi^{\frac{m}{2}}a\left(-\frac{m}{2}\right).$$

Multi-dim

Direct computation shows:

$$\begin{aligned} I &= 2^m \int_{\mathbb{R}_+^m} \left[\sum_{l=0}^{\infty} \phi_l a(l) \left(x_1^2 + \cdots + x_m^2 \right)^l \right] dx_1 \cdots dx_m \\ &= 2^m \int_{\mathbb{R}_+^m} \sum_{l=0}^{\infty} \phi_l a(l) \sum_{\substack{n_1,\ldots,n_m \\ n_1+\cdots+n_m=l}} \binom{l}{n_1, \cdots, n_m} x_1^{2n_1} \cdots x_m^{2n_m} dx_1 \cdots dx_m \\ &= 2^m \sum_{l=n_1+\cdots+n_m} \phi_l a(l) \sum_{n_1,\ldots,n_m} \phi_{1,\ldots,m} \binom{l}{n_1, \cdots, n_m} \frac{1}{\phi_{1,\ldots,m}} \prod_{j=1}^{m} \langle 2n_j + 1 \rangle \\ &= AC\ldots \\ &= \pi^{\frac{m}{2}} a\left(-\frac{m}{2} \right), \end{aligned}$$

as desired.

Multi-dim

$$I=\int_{\mathbb{R}_+^m}\frac{x_1^{p_1-1}\cdots x_m^{p_m-1}dx_1\cdots dx_m}{\left(r_0+r_1x_1+\cdots+r_mx_m\right)^s}=\frac{\Gamma\left(p_1\right)\cdots\Gamma\left(p_m\right)\Gamma\left(s-p_1-p_2-\cdots-p_n\right)}{r_1^{p_1}\cdots r_m^{p_m}r_0^{s-p_1-\cdots-p_m}\Gamma\left(s\right)}.$$

$$\left(r_0+r_1x_1+\cdots+r_mx_m\right)^{-s}=\sum_{n_0,n_1,\ldots,n_m}\phi_{0,1,\ldots,m}r_0^{n_0}r_1^{n_1}x_1^{n_1}\cdots r_m^{n_m}x_m^{n_m}\frac{\langle s+n_0+\cdots+n_m\rangle}{\Gamma\left(s\right)}$$

$$I=\frac{1}{\Gamma\left(s\right)}\sum_{n_0,n_1,\ldots,n_m}\phi_{0,1,\ldots,m}r_0^{n_0}\cdots r_m^{n_m}\left\langle s+n_0+\cdots+n_m\right\rangle\prod_{j=1}^{m}\left\langle n_m+p_m\right\rangle.$$

$$\begin{bmatrix}1&1&1&\cdots&1\\0&1&0&\cdots&0\\\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&0&0&1\end{bmatrix}\begin{bmatrix}n_0\\n_1\\\cdots\\n_m\end{bmatrix}+\begin{bmatrix}s\\p_1\\\cdots\\p_m\end{bmatrix}=\begin{bmatrix}0\\0\\0\\\cdots\\0\end{bmatrix},$$

$$\det A=1,\ n_j^*=-p_j,\forall j=1,\ldots,m\ \text{ and }\ n_0^*=p_1+\cdots+p_m-s.$$

Null/Divergent Series

$K_0(x) = \int_0^\infty \frac{\cos(tx)dt}{\sqrt{1+t^2}}$.

$$K_0(x) = \frac{1}{2}\sum_n \phi_n \Gamma(-n) \frac{x^{2n}}{4^n} \text{ and } K_0(x) = \sum_n \phi_n \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(-n)} \cdot \frac{4^n}{x^{2n+1}}$$

$$\mathcal{M}(K_0)(s) = \int_0^\infty x^{s-1} K_0(x)\,dx = 2^{s-2}\Gamma\left(\frac{s}{2}\right)^2$$

$$\begin{aligned}
&\int_0^\infty x^{s-1} K_0(x)\,dx \\
=& \int_0^\infty x^{s-1} \sum_n \phi_n \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(-n)} \cdot \frac{4^n}{x^{2n+1}}\,dx \\
=& \sum_n \phi_n \frac{\Gamma\left(n+\frac{1}{2}\right)^2 4^n}{\Gamma(-n)} \langle s-2n-1\rangle \\
=& 2^{s-2}\Gamma\left(\frac{s}{2}\right)^2,
\end{aligned}$$

$$\begin{aligned}
&\int_0^\infty x^{s-1} K_0(x)\,dx \\
=& \int_0^\infty \frac{x^{s-1}}{2} \sum_n \phi_n \Gamma(-n) \frac{x^{2n}}{4^n}\,dx \\
=& \frac{1}{2}\sum_n \phi_n \frac{\Gamma(-n)}{4^n} \langle 2n+s\rangle \\
=& 2^{s-2}\Gamma\left(\frac{s}{2}\right)^2.
\end{aligned}$$

DEF

A. Kempf et al. study Dirac-delta function to obtain the following formulas:

$$
\begin{aligned}
\int_a^b f(x)\,dx &= \lim_{\varepsilon\to 0} f(\partial_\varepsilon)\frac{e^{\varepsilon b}-e^{\varepsilon a}}{\varepsilon}, \\
\int_{-\infty}^{\infty} f(x)\,dx &= \lim_{\varepsilon\to 0} 2\pi f(-\iota\partial_\varepsilon)\,\delta(\varepsilon) = 2\pi\delta(\iota\partial_\varepsilon)\,f(\varepsilon), \\
\int_0^{\infty} f(x)\,dx &= \lim_{\varepsilon\to 0} f(-\partial_\varepsilon)\frac{1}{\varepsilon}, \\
\int_{-\infty}^{0} f(x)\,dx &= \lim_{\varepsilon\to 0} f(\partial_\varepsilon)\frac{1}{\varepsilon}, \\
\int_{-\infty}^{\infty} f(x)\,dx &= \lim_{\varepsilon\to 0} \left[f(-\partial_\varepsilon)+f(\partial_\varepsilon)\right]\frac{1}{\varepsilon},
\end{aligned}
$$

where ∂_ε denotes the derivative with respect to ε.

Example

$$I=\int_0^\infty \frac{\sin x}{x}dx=\frac{\pi}{2}$$

$$f(x)=\frac{1}{x}\cdot\frac{1}{2\iota}\left(e^{\iota x}-e^{-\iota x}\right)$$

$$I=\lim_{\varepsilon\to 0} f(-\partial_\varepsilon)\frac{1}{\varepsilon}=\frac{1}{2\iota}\lim_{\varepsilon\to 0}\left(e^{-\iota\partial_\varepsilon}-e^{\iota\partial_\varepsilon}\right)\frac{1}{\partial_\varepsilon}\circ\frac{1}{\epsilon}.$$

Note that $1/\partial_\varepsilon$ is the inverse operation of derivative, i.e., integration.

$$I=\frac{1}{2\iota}\lim_{\varepsilon\to 0}\left(e^{-\iota\partial_\varepsilon}-e^{\iota\partial_\varepsilon}\right)\circ(\ln\varepsilon+c)$$

Recall that for the derivative operator ∂_ε, so that

$$e^{a\partial_\varepsilon}\circ f(\varepsilon)=f(\varepsilon+a).$$

$$\begin{aligned} I &= \frac{1}{2\iota}\lim_{\varepsilon\to 0}\left[(\ln(\varepsilon-\iota)+c)-(\ln(\varepsilon+\iota)+c)\right] \\ &= \frac{1}{2\iota}\lim_{\varepsilon\to 0}\left[\ln(\varepsilon-\iota)-\ln(\varepsilon+\iota)\right]=\frac{1}{2\iota}\left(\frac{-\iota\pi}{2}-\frac{\iota\pi}{2}\right)=\frac{\pi}{2}. \end{aligned}$$

Remark

$$
\begin{aligned}
I &= \int_0^\infty \frac{1}{x} \sum_n \phi_n \frac{\Gamma(n+1)}{\Gamma(2n+2)} x^{2n+1} dx \\
&= \sum_n \phi_n \frac{\Gamma(n+1)}{\Gamma(2n+2)} \langle 2n+1 \rangle \\
&= \frac{1}{2} \cdot \frac{\Gamma\left(-\frac{1}{2}+1\right)}{\Gamma\left(2\left(-\frac{1}{2}\right)+2\right)} \Gamma\left(\frac{1}{2}\right) \\
&= \frac{\pi}{2}.
\end{aligned}
$$

Proofs

D. Jia , E. Tang, A. Kempf, "present a list of propositions that put the above integration by differentiation methods on a rigorous footing."

$$\int_0^\infty f(x)\, e^{-xy}\, dx = \lim_{a\to\infty} f(-\partial_y)\, \frac{1-e^{-ay}}{y},$$

provided that $f:\mathbb{R}\to\mathbb{R}$ is entire and Laplace transformable on $\mathbb{R}_+$. Formal/Key idea:

$$\begin{aligned}
\int_0^\infty f(x)\, e^{-xy}\, dx &= \int_0^\infty \sum_{n=0}^\infty c_n x^n e^{-xy}\, dx \\
&= \sum_{n=0}^\infty c_n \lim_{a\to\infty} \int_0^a x^n e^{-xy}\, dx \\
&= \sum_{n=0}^\infty c_n \lim_{a\to\infty} \int_0^a (-\partial_y)^n\, e^{-xy}\, dx.
\end{aligned}$$

Formal Connection

$$f(x)=\sum_{n=0}^{\infty}a_n x^{\alpha n+\beta-1}\Rightarrow\int_0^{\infty}f(x)\,dx=\lim_{\varepsilon\to 0}\int_0^{\infty}e^{-\varepsilon x}f(x)\,dx.$$

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_0^{\infty}e^{-\varepsilon x}f(x)\,dx &= \lim_{\varepsilon\to 0}\int_0^{\infty}e^{-\varepsilon x}\left(\sum_{n=0}^{\infty}a_n x^{\alpha n+\beta-1}\right)dx\\ &= \lim_{\varepsilon\to 0}\sum_{n=0}^{\infty}a_n\int_0^{\infty}e^{-\varepsilon x}x^{\alpha n+\beta-1}dx\\ &= \lim_{\varepsilon\to 0}\sum_{n=0}^{\infty}a_n\int_0^{\infty}\left((-\partial_\varepsilon)^{\alpha n+\beta-1}\circ e^{-\varepsilon x}\right)dx\\ &= \lim_{\varepsilon\to 0}\sum_{n=0}^{\infty}a_n(-\partial_\varepsilon)^{\alpha n+\beta-1}\circ\int_0^{\infty}e^{-\varepsilon x}dx\\ &= \lim_{\varepsilon\to 0}f(-\partial_\varepsilon)\circ\frac{1}{\varepsilon}.\end{aligned}$$

Formal Connection

Recall P_1:

$$f(x)=\sum_{n=0}^{\infty}a_n x^{\alpha n+\beta-1}\Rightarrow\int_0^{\infty}f(x)\,dx=\sum_n a_n\langle\alpha n+\beta\rangle.$$

Formally,

$$\langle a\rangle:=\int_0^{\infty}x^{a-1}dx.$$

and

$$\langle a\rangle_{\varepsilon}:=\int_0^{\infty}x^{a-1}e^{-\varepsilon x}dx\Rightarrow\lim_{\varepsilon\to0}\langle a\rangle_{\varepsilon}=\langle a\rangle.$$

Therefore

$$\begin{aligned}\sum_n a_n\langle\alpha n+\beta\rangle &= \lim_{\varepsilon\to0}\sum_n a_n\langle\alpha n+\beta\rangle_{\varepsilon}\\ &= \lim_{\varepsilon\to0}\sum_n a_n\int_0^{\infty}x^{\alpha n+\beta-1}e^{-\varepsilon x}dx\\ &= \lim_{\varepsilon\to0}\int_0^{\infty}e^{-\varepsilon x}f(x)\,dx.\end{aligned}$$

Possible Future Work

- Connect MoB to IbD, and provide rigorous proofs of the former.
- Ramanujan's Master Theorem.
- Extension to other intervals.
- etc.

References

B. C. Berndt, *Ramanujan's Notebooks Part I*, Springer-Verlag, 1991.

K. Boyadzhiev, V. H. Moll. The integrals in Gradshteyn and Ryzhik. Part 21: hyperbolic functions. *Scientia*. **22** (2013), 109–127, 2013.

I. Gonzalez, K. Kohl, and V. H. Moll. Evaluation of entries in Gradshteyn and Ryzhik employing the method of brackets. *Scientia*, **25** (2014), 65–84.

I. Gonzalez and V. H. Moll. Definite integrals by the method of brackets. Part 1. *Adv. Appl. Math.*, **45** (2010), 50–73.

I. Gonzalez and I. Schmidt. Optimized negative dimensional integration method (NDIM) and multiloop Feynman diagram calculation. *Nuclear Phys. B*, **769** (2017), 124–173.

I. Gonzalez and I. Schmidt. Modular application of an integration by fractional expansion (IBFE) method to multiloop Feynman diagrams. *Phys. Rev. D*, **78** (2008), 086003.

I. G. Halliday and R. M. Ricotta. Negative dimensional integrals. I. Feynman graphs. *Phys. Lett. B*, **193** (1987), 241–246.

I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products*, Academic Press, 2015.

G. H. Hardy. *Ramanujan. Twelve Lectures on Subjects Suggested by His Life and Work*. Chelsea Publishing Company, 1987.

K. Kohl. *Algorithmic Methods for Definite Integration*. PhD thesis, Tulane University, 2011.

References

A. Kempf, D. M. Jackson, and A. H. Morales, *New Dirac Delta Function Based Methods with Applications to Perturbative Expansions in Quantum Field Theory*, Journal of Physics A: Mathematical and Theoretical 47 (41): 415-204, 2014

A. Kempf, D. M. Jackson, and A. H. Morales, *How to (Path-) Integrate by Differentiating*, preprint, arXiv:math/1507.04348, 2015.

D. Jia, E. Tang and A. Kempf, *Integration by differentiation: new proofs, methods and example*s, preprint, https://arxiv.org/abs/1610.09702, 2016.

End

Thank you!