On Bernoulli Symbol $\mathcal{B}$

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Bernoulli Numbers & Bernoulli Polynomials

Definition

Bernoulli numbers B_n and Bernoulli polynomials $B_n(x)$ are given by their exponential generating functions:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad \text{and} \quad \frac{t e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

Example

- Faulhaber's formula:

$$1^n + 2^n + \cdots + N^n = \frac{1}{n+1} \sum_{i=1}^{n+1} \binom{n+1}{i} B_{n+1-i} N^i = \frac{B_{n+1}(N+1) - B_{n+1}}{n+1}.$$

- Riemann-zeta:

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n} B_{2n}}{2(2n)!}, \quad \zeta(-n) = -\frac{B_{n+1}}{n+1}.$$

Umbral Calculus

Key Idea:

$$\mathcal{B}^n \mapsto B_n : \text{ i.e., super index} \leftrightarrow \text{lower index.}$$

Why?

Simplification

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k = (\mathcal{B} + x)^n .$$

Simplification

Faulhaber's formula:

$$\begin{aligned} 1^n + \cdots + N^n &= \frac{B_{n+1}(N+1) - B_{n+1}}{n+1}. \\ &= \frac{1}{n+1}\left((\mathcal{B} + N + 1)^{n+1} - \mathcal{B}^{n+1} \right) \\ &= \left(\Delta_{N+1} \cdot \int_0^t \right) (\mathcal{B} + x)^n \, dx \Big|_{t=0} . \end{aligned}$$

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Umbral Calculus (Cont.)

Visualization

$$B_n'(x) = nB_{n-1}(x) \Leftrightarrow \left[(\mathcal{B}+x)^n\right]' = n(\mathcal{B}+x)^{n-1}.$$

New Aspect (Probabilitistic Interpretation)

$\exists p(t)$ on $\mathbb{R}$ s. t. (moment)

$$\mathcal{B}^n = \int_{\mathbb{R}} t^n p(t)\, dt.$$

Umbral Calculus (Cont.)

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Probabilitistic Interpretation

Theorem[Density of $\mathcal{B}$](A. Dixit, V. H. Moll, and C. Vignat)

$\mathcal{B} \sim \iota L_B - \frac{1}{2}$, where

$$\iota^2 = -1,\ L_B \text{ has density } \frac{\pi}{2}\operatorname{sech}^2(\pi t) \text{ on } \mathbb{R}$$

Probabilitistic Interpretation

- $$B_n = \mathcal{B}^n = \mathbb{E}[\mathcal{B}^n] = \frac{\pi}{2}\int_{\mathbb{R}}\left(\iota t - \frac{1}{2}\right)^n \operatorname{sech}^2(\pi t)\,dt.$$
- $$B_n(x) = (\mathcal{B}+x)^n = \frac{\pi}{2}\int_{\mathbb{R}}\left(x + \iota t - \frac{1}{2}\right)^n \operatorname{sech}^2(\pi t)\,dt.$$

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- $$B_n(x) = (\mathcal{B} + x)^n = \frac{\pi}{2}\int_{\mathbb{R}} \left(x + \iota t - \frac{1}{2}\right)^n \operatorname{sech}^2(\pi t)\, dt.$$

Probabilitistic Interpretation (Cont.)

Definition

The uniform symbol $\mathcal{U}$ is defined by the uniform distribution on $[0,1]$, with evaluation/expectation:

$$\mathcal{U}^n = \mathbb{E}\left[\mathcal{U}^n\right] = \int_0^1 t^n dt = \frac{1}{n+1}.$$

Remark

$$e^{\mathcal{B}y} = \mathbb{E}\left[e^{\mathcal{B}y}\right] = \sum_{n=0}^{\infty} B_n \frac{y^n}{n!} = \frac{y}{e^y - 1} \text{and } e^{\mathcal{U}y} = \frac{e^y - 1}{y}.$$

$$e^{(\mathcal{B}+\mathcal{U})y} = e^{\mathcal{B}y} \cdot e^{\mathcal{U}y} = 1 \Rightarrow (\mathcal{B}+\mathcal{U})^n = \delta_{n,0}.$$

$\mathcal{B}$ and $\mathcal{U}$

Difference

For polynomial $P_n(x) = \sum_{k=0}^{n} a_k x^n$

$$P_n(x+\mathcal{B}+\mathcal{U}) = \sum_{k=0}^{n} a_k (x+\mathcal{B}+\mathcal{U})^n = \sum_{k=0}^{n} a_k x^n = P_n(x).$$

Now let $P_{n-1}(x) = x^{n-1}$, we have

$$\begin{aligned} x^{n-1} &= (x+\mathcal{B}+\mathcal{U})^{n-1} = \int_0^1 (x+\mathcal{B}+u)^{n-1}\, du \\ &= \frac{1}{n}\left((x+\mathcal{B}+1)^n - (x+\mathcal{B})^n\right) \\ \Rightarrow &\quad B_n(x+1) - B_n(x) = nx^{n-1}. \end{aligned}$$

Definitions

- Bernoulli:

$$\frac{t}{e^t-1}e^{tx}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\Leftrightarrow B_n(x)=(\mathcal{B}+x)^n$$

- Norlünd:

$$\left(\frac{t}{e^t-1}\right)^p e^{tx}=\sum_{n=0}^{\infty}B_n^{(p)}(x)\frac{t^n}{n!}\Leftrightarrow B_n^{(p)}=\left(\underbrace{\mathcal{B}_1+\cdots+\mathcal{B}_p}_{\text{i. i. d.}}+x\right)^n$$

- Bernoulli-Barnes: for $\mathbf{a}=(a_1,\dots,a_p)$, $|\boldsymbol{a}|=\prod_{l=1}^{p}a_l\neq 0$

$$e^{tx}\prod_{i=1}^{p}\frac{t}{e^{a_i t}-1}=\sum_{n=0}^{\infty}B_n(\mathbf{a};x)\frac{t^n}{n!}\Leftrightarrow B_n(\mathbf{a};x)=\frac{1}{|\mathbf{a}|}\left(x+\mathbf{a}\cdot\vec{\mathcal{B}}\right)^n$$

Norlünd: β symbol

Definition

$$\left(\frac{t}{e^t-1}\right)^p e^{tx} = \sum_{n=0}^{\infty} B_n^{(p)}(x)\frac{t^n}{n!} \Leftrightarrow B_n^{(p)} = (\mathcal{B}_1+\cdots+\mathcal{B}_p+x)^n.$$

Theorem [Lucas Formula (1878)]

$$B_n^{(p+1)} = \left(1-\frac{n}{p}\right)B_n^{(p)} - nB_{n-1}^{(p)} = (-1)^p p\binom{n}{p}\beta^{n-p}(\beta)_p,$$

where $(\beta)_p = \beta(\beta+1)\cdots(\beta+p-1)$ is the Pochhammer symbol and

$$\beta^n = \frac{B_n}{n}.$$

MZV: $\mathcal{C}$ symbol

Definition

$$\zeta_r(n_1,\ldots,n_r) = \sum_{0<k_1<\cdots<k_r} \frac{1}{k_1^{n_1}\cdots k_r^{n_r}} = \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{1}{k_1^{n_1}(k_1+k_2)^{n_2}\cdots(k_1+\cdots+k_r)^{n_r}}$$

B. Sadaoui's analytic continuation is based on Raabe's identity by linking

$$Y_{\mathbf{a}}(\mathbf{n}) = \int_{[1,\infty)^r} \frac{d\mathbf{x}}{(x_1+a_1)\cdots(x_1+a_1+\cdots+x_r+a_r)^{n_r}}$$

to the multiple zeta function

$$Z(\mathbf{n},\mathbf{z}) = \sum_{k_1,\ldots,k_r>0} \frac{1}{(k_1+z_1)^{n_1}\cdots(k_1+z_1+\cdots+k_r+z_r)^{n_r}}$$

MZV: $\mathcal{C}$ symbol (Cont.)

Theorem(Sadaoui)

$$\zeta_r(-n_1,\ldots,-n_r) = (-1)^r \sum_{k_2,\ldots,k_r} \frac{1}{\bar{n}+r-\bar{k}} \prod_{j=2}^{r} \frac{\binom{\sum_{i=j}^{r} n_i - \sum_{i=j+1}^{n} k_i + r - j + 1}{k_j}}{\sum_{i=j}^{r} n_i - \sum_{i=j}^{n} k_i + r - j + 1}$$

$$\times \sum_{l_1,\ldots,l_r} \binom{\bar{n}+r-\bar{k}}{l_1}\binom{k_2}{l_2}\cdots\binom{k_r}{l_r} B_{l_1}\cdots B_{l_r},$$

$\bar{n} = \sum_{j=1}^{n} n_j$, $\bar{k} = \sum_{j=2}^{r} k_j$, $k_2,\ldots k_r \geq 0$, $l_j \leq k_j$ for $2 \leq j \leq r$ and $l_1 \leq \bar{n}+r+\bar{k}$.

Theorem(L. Jiu, V. H. Moll and C. Vignat)

$$\zeta_r(-n_1,\ldots,-n_r) = \prod_{k=1}^{r} (-1)^{n_k} \mathcal{C}_{1,\ldots,k}^{n_k+1},$$

where $\mathcal{C}_1^n = \frac{\mathcal{B}_1^n}{n}$, $\mathcal{C}_{1,2}^n = \frac{(\mathcal{C}_1+\mathcal{B}_2)^n}{n}, \ldots, \mathcal{C}_{1,\ldots,k+1}^n = \frac{(\mathcal{C}_{1,\ldots,k}+\mathcal{B}_{k+1})^n}{n}$

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MZV: $\mathcal{C}$ symbol (Cont.)

Example

$$\begin{aligned}
\zeta_2(-n,0) &= (-1)^n\,\mathcal{C}_1^{n+1}\cdot(-1)^0\,\mathcal{C}_{1,2}^{0+1}\\
&= (-1)^n\,\frac{\mathcal{C}_1+\mathcal{B}_2}{1}\cdot\mathcal{C}_1^{n+1}\\
&= (-1)^n\left(\mathcal{C}_1^{n+2}+\mathcal{B}_2\mathcal{C}_1^{n+1}\right)\\
&= (-1)^n\left[\frac{B_{n+2}}{n+2}-\frac{1}{2}\,\frac{B_{n+1}}{n+1}\right].
\end{aligned}$$

$$\zeta_r(-n_1,\ldots,-n_r,z_1,\ldots,z_r)=\prod_{k=1}^{r}(-1)^{n_k}\,\mathcal{C}_{1,\ldots,k}^{n_k+1}(z_1,\ldots,z_k)$$

We have results on *recurrence*, *generating functions*, *quasi-shuffle identities*.

MZV: $\mathcal{C}$ symbol (Cont.)

Example

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&= (-1)^n \left(\mathcal{C}_1^{n+2} + \mathcal{B}_2 \mathcal{C}_1^{n+1}\right) \\
&= (-1)^n \left[\frac{B_{n+2}}{n+2} - \frac{1}{2}\frac{B_{n+1}}{n+1}\right].
\end{aligned}$$

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MZV: Another Approach

S. Akiyama and Y. Tanigawa uses Euler-Maclaurin summation formula to derive (the recurrence)

$$\begin{aligned}\zeta_r(-n_1,\dots,-n_r) &= -\frac{\zeta_{r-1}(-n_1,\dots,-n_{r-2},-n_{r-1}-n_r-1)}{1+n_r} \\ &\quad -\frac{\zeta_{r-1}(-n_1,\dots,-n_{r-2},-n_{r-1}-n_r)}{2} \\ &\quad +\sum_{q=1}^{n_r}(-n_r)_q\, a_q \zeta_{r-1}(-n_1,\dots,-n_{r-2},-n_{r-1}-n_r+q),\end{aligned}$$

where $a_q = B_{q+1}/(q+1)!$.

Remark

$$B_1 = -\frac{1}{2} \text{ and } (-n)_{-1} = -\frac{1}{n+1}$$

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Hamburger moment problem

Theorem[Density of $\mathcal{B}$](A. Dixit, V. H. Moll, and C. Vignat)

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Question: Whether sech^2 is unique. (Hamburger $\leftrightarrow \mathbb{R}$)
Answer: Yes, (thank to Prof. K. Dilcher)

$$b_n \le \frac{2n!}{(2\pi)^n}$$

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Jacobi Sequence

$L_B \sim p(t) = \frac{\pi}{2}\operatorname{sech}^2(\pi t)$ gives a measure on $\mathbb{R}$: $d\mu(t) = p(t)\,dt$. Then, it is natural to consider the corresponding orthogonal polynomial sequence (OPS) $\{P_n(x)\}_{n=-1}^{\infty}$(leading coefficient 1):

$$P_{n+1}(x) = xP_n(x) - \omega_n P_{n-1}(x) . P_{-1}(x) = 0,\ P_0(x) = 1.$$

Sequence $\{\omega_n\}_{n=0}^{\infty}$ is called the Jacobi sequence.

- Existence of OPS; $\checkmark$
- Computation of ω_n. **Conjecture**. [Mathematica]

$$\omega_n = \frac{n^4}{4(2n-1)(2n+1)}.$$

However, it is just a simple tranform of a known result.

Perhaps, evaluation of the following series is interesting:

$$\sum_{n=0}^{\infty} \mathcal{B}^{\frac{n}{k}} \frac{x^{\frac{n}{k}}}{n!} = e^{(\mathcal{B}x)^{\frac{1}{k}}}$$

Thank you