Bernoulli symbol on multiple zeta values at negative integers

Lin JIU

RICAM, Austrian Science of Academy

ACA'2017 Session: Algorithmic Combinatorics

RICAM
JOHANN·RADON·INSTITUTE
FOR COMPUTATIONAL AND APPLIED MATHEMATICS

JCT
LEV ACADEMIC CENTER

Acknowledgement

Joint Work with

Prof. Victor H. Moll

Prof. Christophe Vignat

Outline

1. Bernoulli Numbers, Polynomials, Symbol
   - Bernoulli numbers and Bernoulli polynomials
   - Bernoulli Symbol $\mathcal{B}$
2. Multiple Zeta Values
   - Definitions and analytic continuation
   - Generalized Bernoulli symbol :$\mathcal{C}$
3. An Interesting Result

Bernoulli Numbers & Bernoulli Polynomials

Definition

Bernoulli numbers B_n and Bernoulli polynomials $B_n(x)$ are given by their exponential generating functions: $(B_{2n+1} = 0)$

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \text{ and } \frac{t e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

Examples

$$1^n + 2^n + \cdots + N^n = \frac{1}{n+1} \sum_{i=1}^{n+1} \binom{n+1}{i} B_{n+1-i} N^i = \frac{B_{n+1}(N+1) - B_{n+1}}{n+1}.$$

Riemann-zeta: for $n \in \mathbb{Z}_+$

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n} B_{2n}}{2(2n)!}, \quad \zeta(-n) = -\frac{B_{n+1}}{n+1} = (-1)^n \frac{B_{n+1}}{n+1}.$$

Umbral Calculus

Key Idea:

$\mathcal{B}^n \mapsto B_n$: i.e., super index$\leftrightarrow$lower index.

Why?

Simplification

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k = (\mathcal{B} + x)^n .$$

And

$$\begin{aligned} 1^n + \cdots + N^n &= \frac{B_{n+1}(N+1) - B_{n+1}}{n+1} = \frac{1}{n+1}\left((\mathcal{B} + N + 1)^{n+1} - \mathcal{B}^{n+1}\right) \\ &= \Delta_{N+1} \circ \left(\int_0^t (\mathcal{B} + x)^n \, dx\right)\bigg|_{t=0} \left(= \left(\Delta \cdot \int\right) \circ B_n(x)\right) \end{aligned}$$

Umbral Calculus (Cont.)

Visualization

$$B_n'(x) = nB_{n-1}(x) \Leftrightarrow \left[(\mathcal{B}+x)^n\right]' = n(\mathcal{B}+x)^{n-1}.$$

New Aspect (Probabilitistic Interpretation)

$\exists p(t)$ on $\mathbb{R}$ s. t. (moment)

$$\mathcal{B}^n = B_n = \int_{\mathbb{R}} t^n p(t)\, dt.$$

Theorem[Density of $\mathcal{B}$](A. Dixit, V. H. Moll, and C. Vignat)

$\mathcal{B} \sim \imath L_B - \frac{1}{2}$, where

$$\imath^2 = -1,\ L_B \text{ has density } \frac{\pi}{2}\operatorname{sech}^2(\pi t) \text{ on } \mathbb{R}$$

Umbral Calculus (Cont.)

Visualization

$$B_n'(x) = nB_{n-1}(x) \Leftrightarrow \left[(\mathcal{B}+x)^n\right]' = n(\mathcal{B}+x)^{n-1}.$$

New Aspect (Probabilitistic Interpretation)

$\exists p(t)$ on $\mathbb{R}$ s. t. (moment)

$$\mathcal{B}^n = B_n = \int_{\mathbb{R}} t^n p(t)\, dt.$$

Theorem[Density of $\mathcal{B}$](A. Dixit, V. H. Moll, and C. Vignat)

$\mathcal{B} \sim \imath L_B - \frac{1}{2}$, where

$$\imath^2 = -1,\ L_B \text{ has density } \frac{\pi}{2}\operatorname{sech}^2(\pi t) \text{ on } \mathbb{R}$$

Probabilitistic Interpretation

Probabilitistic Interpretation

- $$B_n = \mathcal{B}^n = \mathbb{E}\left[\mathcal{B}^n\right] = \frac{\pi}{2}\int_{\mathbb{R}}\left(\iota t - \frac{1}{2}\right)^n \operatorname{sech}^2\left(\pi t\right) dt.$$

- $$B_n\left(x\right) = \left(\mathcal{B} + x\right)^n = \frac{\pi}{2}\int_{\mathbb{R}}\left(x + \iota t - \frac{1}{2}\right)^n \operatorname{sech}^2\left(\pi t\right) dt. \left(\frac{t}{e^t - 1} \mid e^{tx}\right)$$

- Norlünd:

$$\left(\frac{t}{e^t - 1}\right)^p e^{tx} = \sum_{n=0}^{\infty} B_n^{(p)}\left(x\right)\frac{t^n}{n!} \Leftrightarrow B_n^{(p)} = \left(\underbrace{\mathcal{B}_1 + \cdots + \mathcal{B}_p}_{\text{i. i. d.}} + x\right)^n$$

- Bernoulli-Barnes: for $\mathbf{a} = \left(a_1, \ldots, a_p\right)$, $\left|\mathbf{a}\right| = \prod_{l=1}^{p} a_l \neq 0$

$$e^{tx}\prod_{i=1}^{p}\frac{t}{e^{a_i t} - 1} = \sum_{n=0}^{\infty} B_n\left(\mathbf{a}; x\right)\frac{t^n}{n!} \Leftrightarrow B_n\left(\mathbf{a}; x\right) = \frac{1}{\left|\mathbf{a}\right|}\left(x + \mathbf{a}\cdot\vec{\mathcal{B}}\right)^n$$

MZV:Defintion

Recall

Riemann-zeta: for $n \in \mathbb{Z}_+$, the AC $\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$.

Definition

$$\zeta_r(n_1, \ldots, n_r) = \sum_{0<k_1<\cdots<k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}} = \sum_{k_1, \ldots, k_r=0}^{\infty} \frac{1}{k_1^{n_1}(k_1+k_2)^{n_2} \cdots (k_1+\cdots+k_r)^{n_r}}$$

B. Sadaoui's analytic continuation is based on Raabe's identity by linking

$$Y_{\mathbf{a}}(\mathbf{n}) = \int_{[1,\infty)^r} \frac{d\mathbf{x}}{(x_1+a_1)\cdots(x_1+a_1+\cdots+x_r+a_r)^{n_r}}$$

to the multiple zeta function

$$Z(\mathbf{n}, \mathbf{z}) = \sum_{k_1, \ldots, k_r>0} \frac{1}{(k_1+z_1)^{n_1}\cdots(k_1+z_1+\cdots+k_r+z_r)^{n_r}}$$

MZV: Analytic Continuation

Theorem(Sadaoui)

$$\zeta_r(-n_1,\ldots,-n_r) = (-1)^r \sum_{k_2,\ldots,k_r} \frac{1}{\bar{n}+r-\bar{k}} \prod_{j=2}^{r} \frac{\binom{\sum_{i=j}^{r} n_i - \sum_{i=j+1}^{n} k_i + r - j + 1}{k_j}}{\sum_{i=j}^{r} n_i - \sum_{i=j}^{n} k_i + r - j + 1}$$

$$\times \sum_{l_1,\ldots,l_r} \binom{\bar{n}+r-\bar{k}}{l_1}\binom{k_2}{l_2}\cdots\binom{k_r}{l_r} B_{l_1}\cdots B_{l_r},$$

$\bar{n} = \sum_{j=1}^{n} n_j$, $\bar{k} = \sum_{j=2}^{r} k_j$, $k_2, \ldots k_r \geq 0$, $l_j \leq k_j$ for $2 \leq j \leq r$ and $l_1 \leq \bar{n} + r + \bar{k}$.

$$B_{l_i} \mapsto \mathcal{B}_i^{l_i}$$

MZV: Analytic Continuation

Theorem(Sadaoui)

$$\zeta_r(-n_1,\ldots,-n_r) = (-1)^r \sum_{k_2,\ldots,k_r} \frac{1}{\bar{n}+r-\bar{k}} \prod_{j=2}^{r} \frac{\binom{\sum_{i=j}^{r} n_i - \sum_{i=j+1}^{n} k_i + r - j + 1}{k_j}}{\sum_{i=j}^{r} n_i - \sum_{i=j}^{n} k_i + r - j + 1}$$

$$\times \sum_{l_1,\ldots,l_r} \binom{\bar{n}+r-\bar{k}}{l_1}\binom{k_2}{l_2}\cdots\binom{k_r}{l_r} B_{l_1}\cdots B_{l_r},$$

$\bar{n} = \sum_{j=1}^{n} n_j$, $\bar{k} = \sum_{j=2}^{r} k_j$, $k_2,\ldots k_r \geq 0$, $l_j \leq k_j$ for $2 \leq j \leq r$ and $l_1 \leq \bar{n}+r+\bar{k}$.

$$B_{l_i} \mapsto \mathcal{B}_i^{l_i}$$

$\mathcal{C}$ symbol

Theorem(L. Jiu, V. H. Moll and C. Vignat)

$$\zeta_r\left(-n_1,\ldots,-n_r\right)=\prod_{k=1}^{r}(-1)^{n_k}\,\mathcal{C}_{1,\ldots,k}^{n_k+1},$$

where $\quad \mathcal{C}_1^n=\dfrac{\mathcal{B}_1^n}{n},\ \mathcal{C}_{1,2}^n=\dfrac{\left(\mathcal{C}_1+\mathcal{B}_2\right)^n}{n},\ldots,\mathcal{C}_{1,\ldots,k+1}^n=\dfrac{\left(\mathcal{C}_{1,\ldots,k}+\mathcal{B}_{k+1}\right)^n}{n}$

Example

$$\zeta\left(-n\right)=(-1)^n\,\mathcal{C}^{n+1}=(-1)^n\,\frac{B_{n+1}}{n+1}.$$

$$\begin{aligned}
\zeta_2\left(-n,0\right) &= (-1)^n\,\mathcal{C}_1^{n+1}\cdot(-1)^0\,\mathcal{C}_{1,2}^{0+1}\\
&= (-1)^n\,\frac{\mathcal{C}_1+\mathcal{B}_2}{1}\cdot\mathcal{C}_1^{n+1}\\
&= (-1)^n\left(\mathcal{C}_1^{n+2}+\mathcal{B}_2\mathcal{C}_1^{n+1}\right)\\
&= (-1)^n\left[\frac{B_{n+2}}{n+2}-\frac{1}{2}\,\frac{B_{n+1}}{n+1}\right].
\end{aligned}$$

Results

Theorem(L. Jiu, V. H. Moll and C. Vignat)

$$\zeta_r\left(n_1, \ldots, n_r; z_1, \ldots, z_r\right) := \sum_{k_1, \ldots, k_r > 0} \frac{1}{\left(k_1 + z_1\right)^{n_1} \cdots \left(k_1 + z_1 + \cdots + k_r + z_r\right)^{n_r}}$$

$$\zeta_r\left(-n_1, \ldots, -n_r; z_1, \ldots, z_r\right) := \prod_{k=1}^{r} (-1)^{n_k} \left(\mathcal{C}_{1,\ldots,k} + z_k\right)^{n_k+1},$$

where $\left(\mathcal{C}_{1,\ldots,k+1} + x\right)^n = \left(\mathcal{C}_{1,\ldots,k} + \mathcal{B}_{k+1} + x\right)^n / n$.

Theorem(L. Jiu, V. H. Moll and C. Vignat)

- Recurrence:

$$\zeta_r(-\mathbf{n};\mathbf{z}) = \frac{(-1)^{n_r}}{n_r+1}\sum_{l=0}^{n_r+1}\binom{n_r+1}{l}(-1)^l\,\zeta_{r-1}(-\mathbf{n}_{r-2},-n_{r-1}-l;\mathbf{z}_{r-1})\,B_{n_1+1-l}(z_r)\,;$$

- Contiguity: for $\mathcal{Z}_r^l = \zeta_r(-\mathbf{n}_{r-1},-n_r-l;\mathbf{z})$:

$$\zeta_r(-\mathbf{n};\mathbf{z}_{r-1},z_r+1) = \zeta_r(-\mathbf{n};\mathbf{z}_{r-1},z_r) + (-1)^{n_r}(z_r-\mathcal{Z}_{r-1})^{n_r}\,;$$

Theorem(L. Jiu, V. H. Moll and C. Vignat)

- Generating Function

$$F_r(w_1, \ldots, w_r) := \sum_{n_1, \ldots, n_r \geq 0} \frac{w_1^{n_1} \cdots w_r^{n_r}}{n_1! \cdots n_r!} \zeta_r(-n_1, \ldots, -n_r)$$
$$= (F_1(w_r, -\partial_{r-1}) \cdots F_1(w_2, -\partial_1)) \bullet F_1(w_1, 0),$$

where $\partial_i = \partial / \partial w_i$ and

$$F_1(w, z) := \sum_{n=0}^{\infty} \frac{w^n}{n!} \zeta(-n, z) = \frac{e^{-wz}}{e^{-w} - 1} - \frac{1}{w}.$$

MZV: Another Approach

S. Akiyama and Y. Tanigawa uses Euler-Maclaurin summation formula to derive (the recurrence)

$$\begin{aligned}\zeta_r\left(-n_1,\ldots,-n_r\right) &= -\frac{\zeta_{r-1}\left(-n_1,\ldots,-n_{r-2},-n_{r-1}-n_r-1\right)}{1+n_r} \\ &\quad -\frac{\zeta_{r-1}\left(-n_1,\ldots,-n_{r-2},-n_{r-1}-n_r\right)}{2} \\ &\quad +\sum_{q=1}^{n_r}\left(-n_r\right)_q a_q\zeta_{r-1}\left(-n_1,\ldots,-n_{r-2},-n_{r-1}-n_r+q\right),\end{aligned}$$

where $a_q = B_{q+1}/\left(q+1\right)!$.

Remark

$$B_1 = -\frac{1}{2} \text{ and } \left(-n\right)_{-1} = -\frac{1}{n+1}$$

MZV: Another Approach

S. Akiyama and Y. Tanigawa uses Euler-Maclaurin summation formula to derive (the recurrence)

$$\begin{aligned}\zeta_r(-n_1,\ldots,-n_r) \quad = \quad & -\frac{\zeta_{r-1}(-n_1,\ldots,-n_{r-2},-n_{r-1}-n_r-1)}{1+n_r} \\ & -\frac{\zeta_{r-1}(-n_1,\ldots,-n_{r-2},-n_{r-1}-n_r)}{2} \\ & +\sum_{q=1}^{n_r}(-n_r)_q\, a_q \zeta_{r-1}(-n_1,\ldots,-n_{r-2},-n_{r-1}-n_r+q),\end{aligned}$$

where $a_q = B_{q+1}/(q+1)!$.

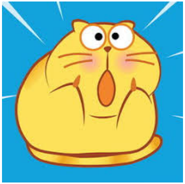
Remark

$$B_1 = -\frac{1}{2} \text{ and } (-n)_{-1} = -\frac{1}{n+1}$$

What's Next

Theorem(L. Jiu, V. H. Moll and C. Vignat)

$$\zeta_r\left(-n_1,\ldots,-n_r\right)=\prod_{k=1}^{r}\left(-1\right)^{n_k}\mathcal{C}_{1,\ldots,k}^{n_k+1}$$

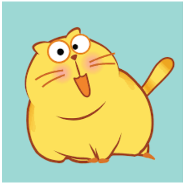


This shows the two appoaches, by Raabe's identity and Euler-Maclaurin summation formula, lead to analytic continuations of MZVs, which coincide on negative integer values.

Why?

Well,.... I do not know.....

What's Next



- Raabe's identity and Euler-Maclaurin summation formula;
- Probabilitistic aspect;
- Bernoulli symbol on other areas;

Thank you