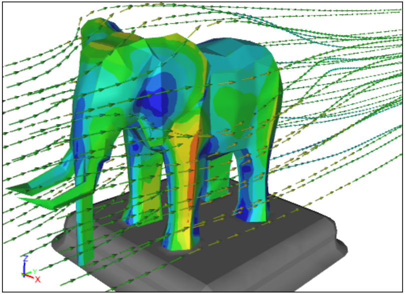
Visualization of Bernoulli Numbers

Lin JIU

October 12, 2017

Outlines

NOT



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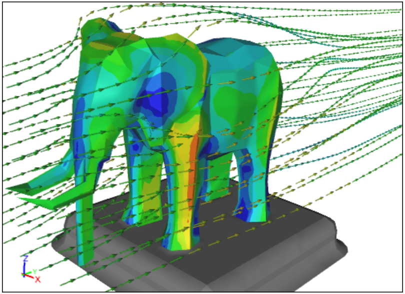

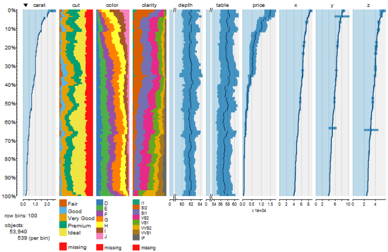
carat
cut
color
clarity
depth
table
price
x
y
z
row bins: 100
objects:
53,940
539 (per bin)
Fair
Good
Very Good
Premium
Ideal
missing
D
E
F
G
H
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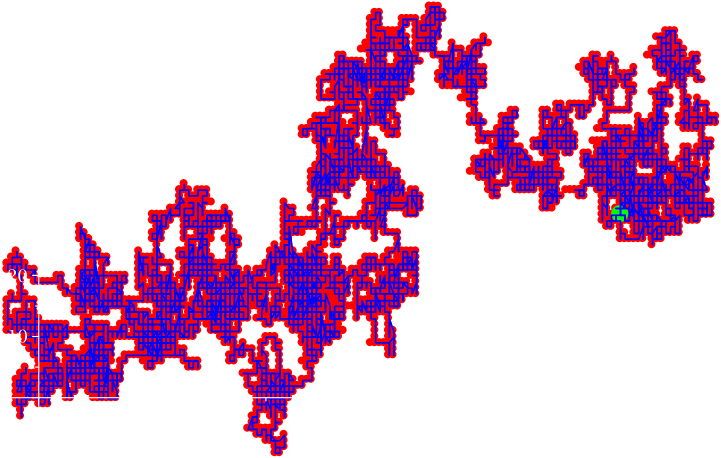
J
missing
I1
SI2
SI1
VS2
VS1
VVS2
VVS1
IF
missing

Game

Game

Round 1

Round 1



Round 1

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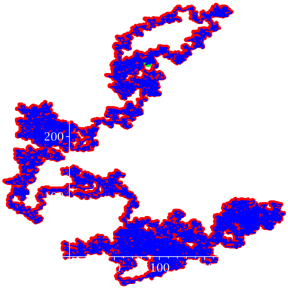

Normal Number

For number n, consider in any base b,

$$n = \underbrace{N_l N_{l-1} \cdots N_1 N_0}_{\text{Integer Part}}.n_1 n_2 n_3 \cdots .$$

Then, for any $a \in \{0, 1, \ldots, b\}$,

$$\frac{1}{b} = \lim_{k \to \infty} \frac{\#\text{ of } a \text{ appearing in } \{N_l, \ldots, N_0, n_1, \ldots, n_k\}}{l + k}.$$

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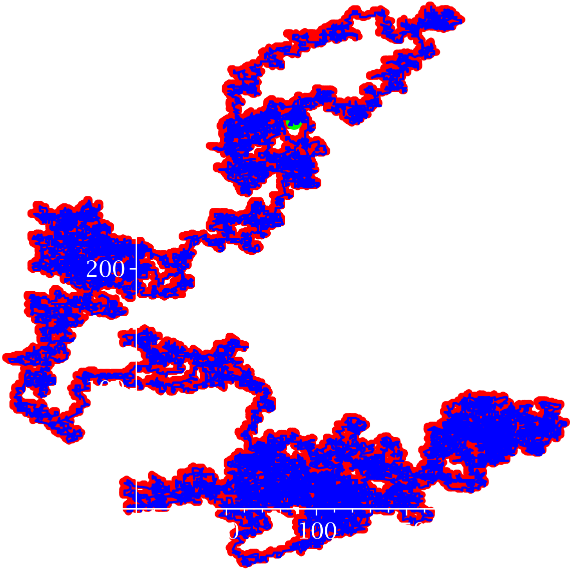
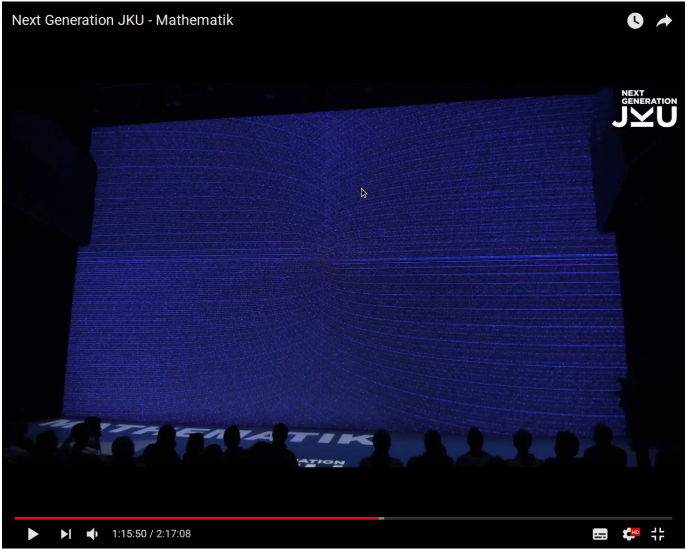
Chaitin's Constant   halting probability

π



Round 2

Round 2


Next Generation JKU - Mathematik
NEXT GENERATION JKU
MATHEMATIK
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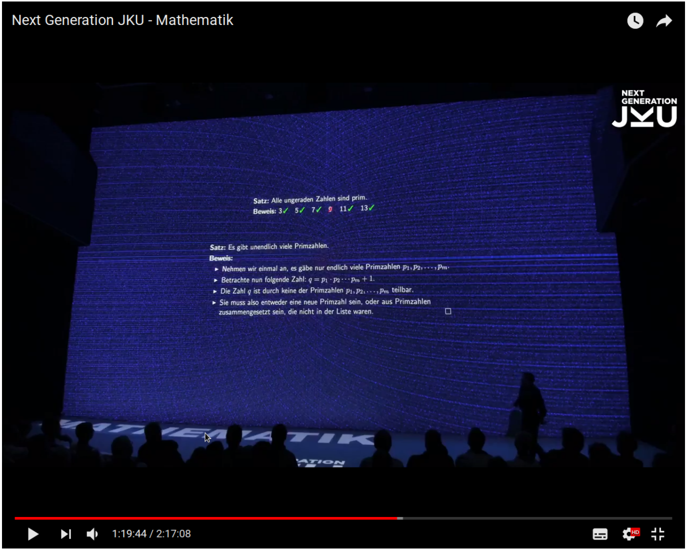
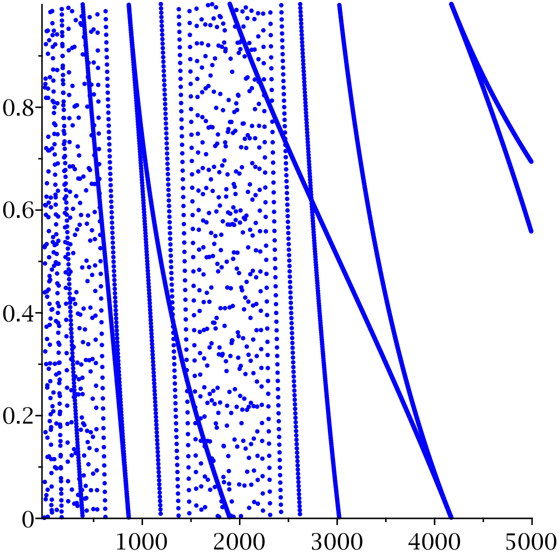
Round 2



Next Generation JKU - Mathematik
NEXT GENERATION JKU
Satz: Alle ungeraden Zahlen sind prim.
Beweis: 3✓ 5✓ 7✓ 9 11✓ 13✓
Satz: Es gibt unendlich viele Primzahlen.
Beweis:
Nehmen wir einmal an, es gäbe nur endlich viele Primzahlen $p_1, p_2, \ldots, p_m$.
Betrachte nun folgende Zahl: $q = p_1 \cdot p_2 \cdots p_m + 1$.
Die Zahl q ist durch keine der Primzahlen $p_1, p_2, \ldots, p_m$ teilbar.
Sie muss also entweder eine neue Primzahl sein, oder aus Primzahlen zusammengesetzt sein, die nicht in der Liste waren. □
1:19:44 / 2:17:08

Round 2.5



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$$x_n := \tan\left(\sum_{k=1}^{n} \tan^{-1}(k)\right) \Rightarrow x_n = \frac{x_{n-1} + n}{1 - n x_{n-1}}.$$

Round 2.5

$$x_1 = 1$$

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$$\begin{aligned} x_1 &= 1 \\ x_2 &= -3 \end{aligned}$$

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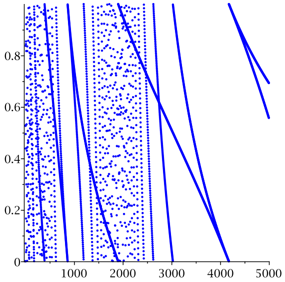

Round 3

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$$1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490,$$
$$627, 792, 1002, 1255, 1575, 1958, 2436, 3010, 3718, 4565, 5604, 6842,$$
$$8349, 10143, 12310, 14883, 17977, 21637, 26015, 31185, 37338, 44583,$$
$$53174, 63261, 75175, 89134, 105558, 124754, 147273, 173525, 204226$$

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Ramanujan's Congruences

$$\begin{aligned} p(5n + 4) &\equiv 0 \pmod{5}, \\ p(7n + 5) &\equiv 0 \pmod{7}, \\ p(11n + 6) &\equiv 0 \pmod{11}. \end{aligned}$$

Borwein Integrals

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Break

Definition

The Bernoulli numbers $(B_n)_{n=0}^{\infty}$ and Bernoulli polynomials $(B_n(x))_{n=0}^{\infty}$ can be defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \text{ and } \frac{t e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

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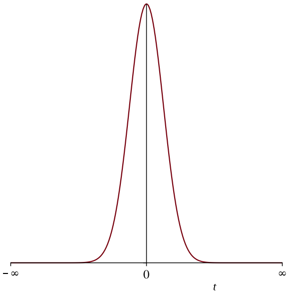
$$B_0=1,\ B_1=-\frac{1}{2},\ B_2=\frac{1}{6},\ B_3=0,\ B_4=-\frac{1}{30},\ldots$$

$$B_0(x)=1,\ B_1(x)=x-\frac{1}{2}\ B_2=-x^2+x+\frac{1}{6},\ldots$$

Round 4



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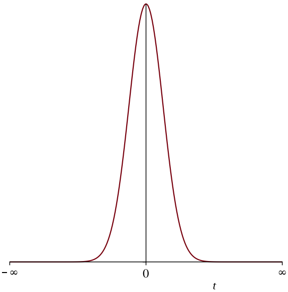

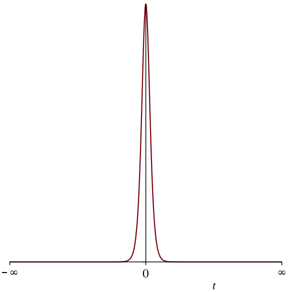

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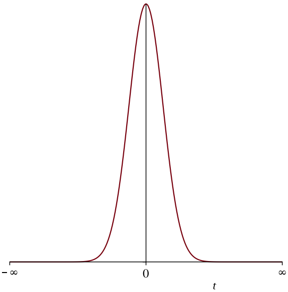

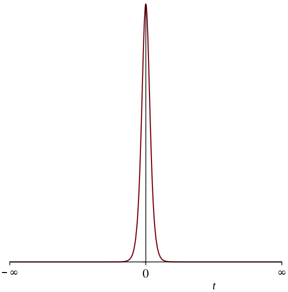

$$p\,(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$$

$$q\,(t) := \frac{\pi}{2} \operatorname{sech}^2(\pi t)$$

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and

$$B_n(x) = \frac{\pi}{2}\int_{\mathbb{R}} \left(x + \imath t - \frac{1}{2}\right)^n \text{sech}^2(\pi t)\,\mathrm{d}t = \mathbb{E}\left[(\mathcal{B} + x)^n\right].$$

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$$B_n(x) = \frac{\pi}{2}\int_{\mathbb{R}} \left(x + \imath t - \frac{1}{2}\right)^n \operatorname{sech}^2(\pi t)\,\mathrm{d}t = \mathbb{E}\left[(\mathcal{B}+x)^n\right].$$

By omitting expectation operator $\mathbb{E}$, we have

$$B_n = \mathcal{B}^n \text{ and } B_n(x) = (\mathcal{B}+x)^n.$$

Bernoulli (Random) Symbol

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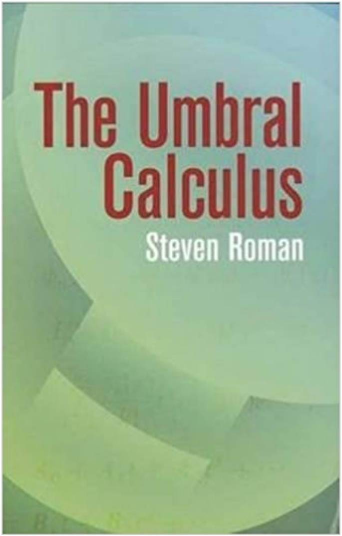
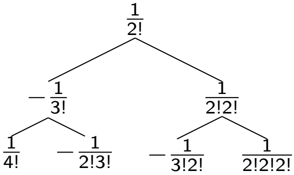
$$B_n'(x) = nB_{n-1}(x) \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}x}(\mathcal{B}+x)^n = n(\mathcal{B}+x)^{n-1}.$$

Umbral Calculus



Woon's Tree



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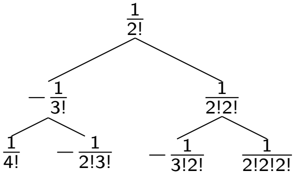

Summing each row gives $\left((-1)^n \frac{B_n}{n!}\right)_{n=1}^{\infty}$.

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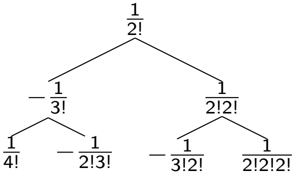

Summing each row gives $\left((-1)^n \frac{B_n}{n!}\right)_{n=1}^{\infty}$.

$$B_1 = -\frac{1}{2} \Rightarrow (-1)^1 \frac{\left(-\frac{1}{2}\right)}{1!} = \frac{1}{2} = \frac{1}{2!};$$

$$B_2 = \frac{1}{6} \Rightarrow (-1)^2 \frac{\frac{1}{6}}{2!} = \frac{1}{12} = \frac{1}{4} - \frac{1}{6} = -\frac{1}{3!} + \frac{1}{2!2!};$$

$$\ldots$$

Faulhaber's formula

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

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$$\begin{aligned} 1 + 2 + \cdots + n &= \frac{n(n+1)}{2} \\ 1^2 + 2^2 + \cdots + n^2 &= \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

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1^k+2^k+\cdots+n^k &= \frac{B_{k+1}(n+1)-B_k}{k+1}.
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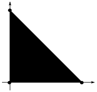
Example

$\frac{B_1(n+1)-B_1(1)}{n+1} = \frac{n(n+1)}{2}$ counts number of integer points in the triangle $(0,0)$, $(0,n)$ and $(n,0)$.



Round ???

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$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}.$$

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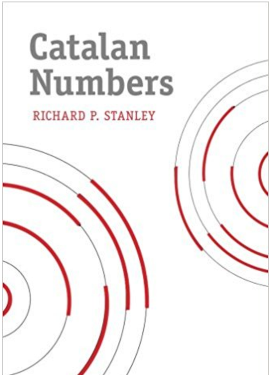

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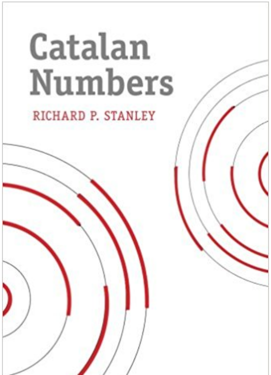
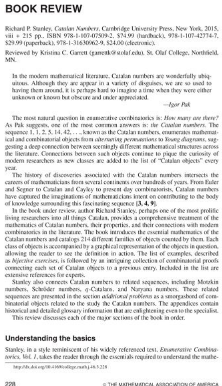




BOOK REVIEW

Richard P. Stanley, *Catalan Numbers*, Cambridge University Press, New York, 2015, viii + 215 pp., ISBN 978-1-107-07509-2, $74.99 (hardback), 978-1-107-42774-7, $29.99 (paperback), 978-1-31630962-9, $24.00 (electronic).

Reviewed by Kristina C. Garrett (garrettk@stolaf.edu), St. Olaf College, Northfield, MN.

> In the modern mathematical literature, Catalan numbers are wonderfully ubiquitous. Although they are appear in a variety of disguises, we are so used to having them around, it is perhaps hard to imagine a time when they were either unknown or known but obscure and under appreciated.
>
> —*Igor Pak*

The most natural question in enumerative combinatorics is: *How many are there?* As Pak suggests, one of the most common answers is: *the Catalan numbers*. The sequence 1, 1, 2, 5, 14, 42, . . ., known as the Catalan numbers, enumerates mathematical and combinatorial objects from *alternating permutations* to *Young diagrams*, suggesting a deep connection between seemingly different mathematical structures across the literature. Connections between such objects continue to pique the curiosity of modern researchers as new classes are added to the list of "Catalan objects" every year.

The history of discoveries associated with the Catalan numbers intersects the careers of mathematicians from several continents over hundreds of years. From Euler and Segner to Catalan and Cayley to present day combinatorists, Catalan numbers have captured the imaginations of mathematicians intent on contributing to the body of knowledge surrounding this fascinating sequence [**3, 4, 9**].

In the book under review, author Richard Stanley, perhaps one of the most prolific living researchers into all things Catalan, provides a comprehensive treatment of the mathematics of Catalan numbers, their properties, and their connections with modern combinatorics in the literature. The book introduces the essential mathematics of the Catalan numbers and catalogs 214 different families of objects counted by them. Each class of objects is accompanied by a graphical representation of the objects in question, allowing the reader to see the definition in action. The list of examples, described as *bijective exercises*, is followed by an intriguing collection of combinatorial proofs connecting each set of Catalan objects to a previous entry. Included in the list are extensive references for experts.

Stanley also connects Catalan numbers to related sequences, including Motzkin numbers, Schröder numbers, q-Catalans, and Naryana numbers. These related sequences are presented in the section *additional problems* as a smorgasbord of combinatorial objects related to the study the Catalan numbers. The appendices contain historical and detailed glossary information that are enlightening even to the specialist.

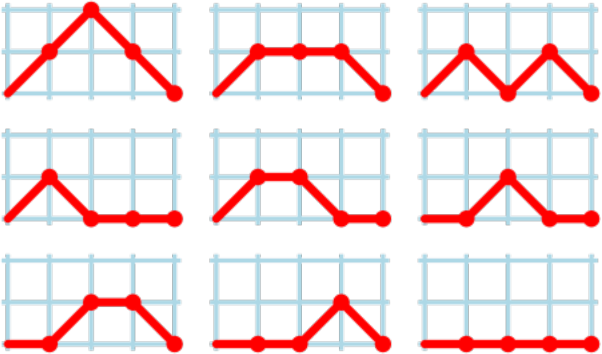
This review discusses each of the major sections of the book in order.

Understanding the basics

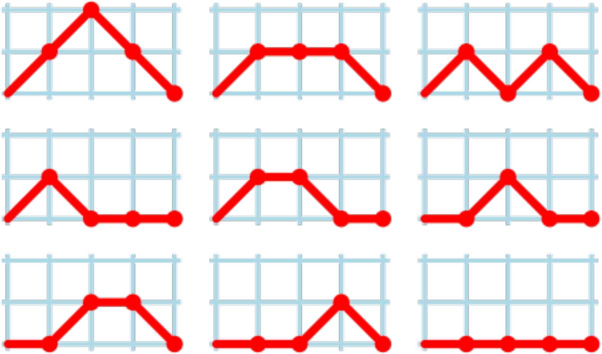
Stanley, in a style reminiscent of his widely referenced text, *Enumerative Combinatorics, Vol. 1*, takes the reader through the essentials required to understand the mathe-

http://dx.doi.org/10.4169/college.math.j.46.3.228

(Generalized) Motzkin Numbers



(Generalized) Motzkin Numbers



$$M_{n+1,k} = M_{n,k-1} + M_{n,k} + M_{n,k+1}$$

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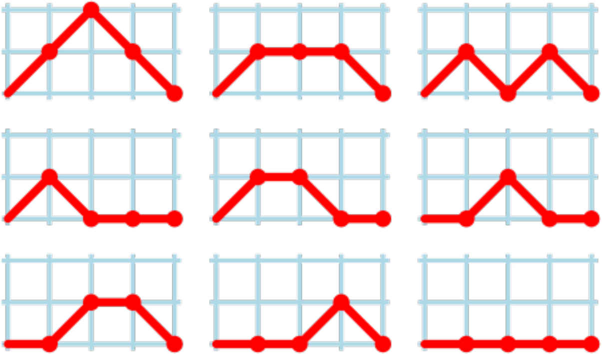

$$M_{n+1,k} = M_{n,k-1} + M_{n,k} + M_{n,k+1}$$

Generalization:

$$M_{n+1,k} = M_{n,k-1} + s_k M_{n,k} + t_{k+1} M_{n,k+1}$$

Bernoulli Numbers

Let $s_k = 1/2$ and $t_k = -\left(k-1\right)^4 / \left(4\left(2k-3\right)\left(2k-1\right)\right)$

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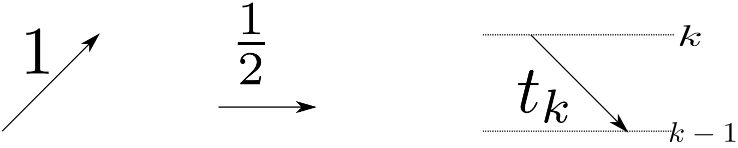

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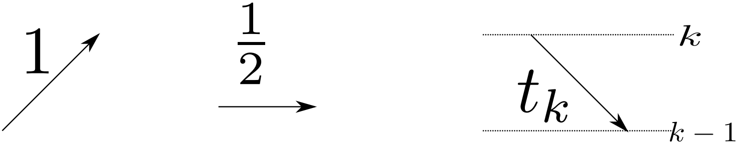

Bernoulli Numbers

$$M = \begin{pmatrix} \frac{1}{2} & 1 & 0 & 0 & 0 & \cdots \\ -\frac{1}{12} & \frac{1}{2} & 1 & 0 & 0 & \cdots \\ 0 & -\frac{4}{15} & \frac{1}{2} & 1 & 0 & \cdots \\ 0 & 0 & -\frac{81}{140} & \frac{1}{2} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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$$M^k(1,1) = B_k.$$

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$$M^k(1,1) = B_k.$$

$$M_4 := \begin{pmatrix} \frac{1}{2} & 1 & 0 & 0 \\ -\frac{1}{12} & \frac{1}{2} & 1 & 0 \\ 0 & -\frac{4}{15} & \frac{1}{2} & 1 \\ 0 & 0 & -\frac{81}{140} & \frac{1}{2} \end{pmatrix} \Rightarrow M_4^4 = \begin{pmatrix} -\frac{1}{30} & -\frac{1}{5} & \frac{4}{7} & 2 \\ \frac{1}{60} & -\frac{13}{70} & -\frac{19}{14} & \frac{4}{7} \\ \frac{4}{315} & \frac{38}{105} & -\frac{689}{1470} & -\frac{25}{21} \\ -\frac{9}{350} & \frac{108}{1225} & \frac{135}{196} & -\frac{31}{98} \end{pmatrix}$$

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$$1, 1, 2, 3, 5, 8, 13, \ldots$$

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$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix}$$

End

Thank you!