Bernoulli Symbol $\mathcal{B}$: from umbral calculus to random variable and combinatorics

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Outlines

Bernoulli Symbol and Umbral Calculus

Probabilistic Aspect

Combinatorial Interpretation

Definition

The Bernoulli numbers $(B_n)_{n=0}^{\infty}$ and Bernoulli polynomials $(B_n(x))_{n=0}^{\infty}$ can be defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \text{ and } \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

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$$\begin{aligned} e^{\mathcal{B}t} = \frac{t}{e^t - 1} \quad &\Rightarrow \quad e^{-\mathcal{B}t} = \frac{-t}{e^{-t}-1} = \frac{te^t}{e^t - 1} = e^{(\mathcal{B}+1)t} \\ &\Rightarrow \quad -\mathcal{B} = \mathcal{B}+1 \end{aligned}$$

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By omitting expectation operator $\mathbb{E}$, we have

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Umbral Calculus

Let $f(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}$, then define a linear functional $\langle \rangle$ on $\mathbb{C}[x]$, by

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$$L(x^n) = B_n$$

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$$L(x^n) = B_n \qquad \mathbb{E}[\mathcal{B}^n] = B_n.$$

Probabilistic Interpretation

Recall: For independent random variables X and Y, if

$$\begin{cases} \mathbb{E}\left[e^{tX}\right] = F(x) & , \\ \mathbb{E}\left[e^{tY}\right] = G(x) & , \end{cases}$$

then

$$\mathbb{E}\left[e^{t(X+Y)}\right] = F(x)\, G(x)\,.$$

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$$B_n(x) = \mathbb{E}\left[(\mathcal{B}+x)^n\right] = \frac{[t^n]\, e^{\mathcal{B}t} e^{xt}}{n!} = \frac{[t^n]\, \frac{te^{xt}}{e^t-1}}{n!}.$$

Generalization

- Bernoulli:
$$\frac{t}{e^t - 1} e^{tx} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$
- Norlünd:
$$\left(\frac{t}{e^t - 1}\right)^p e^{tx} = \sum_{n=0}^{\infty} B_n^{(p)}(x) \frac{t^n}{n!}$$
- Bernoulli-Barnes
$$e^{tx} \prod_{i=1}^{k} \frac{t}{e^{a_i t} - 1} = \sum_{n=0}^{\infty} B_n(\mathbf{a}; x) \frac{t^n}{n!},$$
for $\mathbf{a} = (a_1, \dots, a_k)$.

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- Bernoulli-Barnes ($\forall l = 1, \ldots, n,\ a_l \neq 0$)

$$B_n(\mathbf{a}; x) = \frac{1}{|\mathbf{a}|}\left(x + \mathbf{a} \cdot \vec{\mathcal{B}}\right)^n, \text{ where } \begin{cases} \mathbf{a} = (a_1, \ldots, a_k) \\ \vec{\mathcal{B}} = (\mathcal{B}, \ldots, \mathcal{B}_k) \\ \mathbf{a} \cdot \vec{\mathcal{B}} = \sum_{l=1}^{k} a_l \mathcal{B}_l \\ |\mathbf{a}| = \prod_{l=1}^{k} a_l \end{cases}$$

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This implies

$$\mathbb{E}\left[e^{t(\mathcal{U}+\mathcal{B})}\right] = 1 \Rightarrow (\mathcal{U}+\mathcal{B})^n = \mathbb{E}\left[(\mathcal{U}+\mathcal{B})^n\right] = \delta_{n,0}.$$

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Then, $e^{a\partial}=\Delta_a+I$, i.e,

$$e^{a\partial}\circ f(x)=\sum_{m=0}^{\infty}\frac{a^n\partial^n}{n!}\circ f(x)=\sum_{m=0}^{\infty}\frac{a^nf^{(n)}(x)}{n!}=f(x+a)=(\Delta_a+I)\circ f(x).$$

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$f(x) := x^n$.

Several Results

Bernoulli-Barnes

$$e^{tx}\prod_{i=1}^{k}\frac{t}{e^{a_i t}-1}=\sum_{n=0}^{\infty}B_n\left(\mathbf{a};x\right)\frac{t^n}{n!}\Leftrightarrow B_n\left(\mathbf{a};x\right)=\frac{1}{|\mathbf{a}|}\left(x+\mathbf{a}\cdot\vec{\mathcal{B}}\right)^n$$

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Theorem (A. Bayad and M. Beck)

Difference Formula: Suppose $A=\sum_{k=1}^{n}a_k\neq 0$, *then*

$$(-1)^m B_m(\mathbf{a};-x)-B_m(\mathbf{a};x)=m!\sum_{l=1}^{n-1}\sum_{|L|=l}\frac{B_{m-n+l}(\mathbf{a}_L;x)}{(m-n+l)!},$$

where $L\subset\{1,\ldots,n\}$ *and* $B_m(\mathbf{a}_L;x)=x^m$ *if* $L=\emptyset$.

Several Results

Bernoulli-Barnes

$$e^{tx}\prod_{i=1}^{k}\frac{t}{e^{a_i t}-1}=\sum_{n=0}^{\infty}B_n(\mathbf{a};x)\frac{t^n}{n!}\Leftrightarrow B_n(\mathbf{a};x)=\frac{1}{|\mathbf{a}|}\left(x+\mathbf{a}\cdot\vec{\mathcal{B}}\right)^n$$

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$$f\left(x-\mathbf{a}\cdot\vec{\mathcal{B}}\right)=\sum_{\ell=0}^{n}\sum_{|L|=\ell}|\mathbf{a}|_{L^*}f^{(n-\ell)}\left(x+\left(\mathbf{a}\cdot\vec{\mathcal{B}}\right)_L\right).$$

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$$B_n^{(p+1)} = (-1)^p\, p \binom{n}{p} \beta^{n-p} (\beta)_p\,,$$

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$$\zeta_r(n_1, \ldots, n_r) = \sum_{0<k_1<\cdots<k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}} = \sum_{k_1, \ldots, k_r=1}^{\infty} \frac{1}{k_1^{n_1} (k_1+k_2)^{n_2} \cdots (k_1+\cdots+k_r)^{n_r}}$$

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B. Sadaoui's analytic continuation is based on Raabe's identity by linking

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to the multiple zeta function

$$Z(\mathbf{n}, \mathbf{z}) = \sum_{k_1,\ldots,k_r>0} \frac{1}{(k_1+z_1)^{n_1} \cdots (k_1+z_1+\cdots+k_r+z_r)^{n_r}}$$

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Theorem (Sadaoui)

$$\zeta_r(-n_1, \ldots, -n_r) = (-1)^r \sum_{k_2,\ldots,k_r} \frac{1}{\bar{n}+r-\bar{k}} \prod_{j=2}^{r} \frac{\binom{\sum_{i=j}^{r} n_i - \sum_{i=j+1}^{n} k_i + r - j + 1}{k_j}}{\sum_{i=j}^{r} n_i - \sum_{i=j}^{n} k_i + r - j + 1}$$

$$\times \sum_{l_1,\ldots,l_r} \binom{\bar{n}+r-\bar{k}}{l_1} \binom{k_2}{l_2} \cdots \binom{k_r}{l_r} B_{l_1} \cdots B_{l_r},$$

$\bar{n} = \sum_{j=1}^{n} n_j$, $\bar{k} = \sum_{j=2}^{r} k_j$, $k_2, \ldots k_r \geq 0$, $l_j \leq k_j$ *for* $2 \leq j \leq r$ *and* $l_1 \leq \bar{n} + r + \bar{k}$.

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S. Akiyama and Y. Tanigawa uses Euler-Maclaurin summation formula to derive (the recurrence)

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This shows the two approaches, by Raabe's identity and Euler-Maclaurin summation formula, lead to analytic continuations of MZVs, which coincide on negative integer values.

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Define $b_n := \left|B_n\left(\frac{1}{2}\right)\right|$, then

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Proof.

$$(-1)^{n+1} B_{2n} \sim \frac{2(2n)!}{(2\pi)^{2n}}.$$

□

Lemma

Uniqueness is equivalent to existence of constants C and D, such that

$$|b_n| \le CD^n n!.$$

Cumulants

$$K(t) := \sum_{n=0}^{\infty} \kappa_n \frac{t^n}{n!} = \log \mathbb{E}\left[e^{tX}\right].$$

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Theorem (Faà di Bruno's formula)

For moments $(m_n)_{n=1}^{\infty}$ and cumulants $(\kappa_n)_{n=1}^{\infty}$ it holds that

$$m_n = Y_n(\kappa_1, \ldots, \kappa_n) \text{ and } \kappa_n = \sum_{k=1}^{n} (-1)^{k-1} (k-1)! Y_{n,k}(m_1, \ldots, m_{n-k+1}),$$

where, the partial or incomplete exponential Bell polynomial is given by

$$Y_{n,k}(x_1, \ldots, x_{n-k+1}) := \sum_{\substack{j_1 + \cdots + j_{n-k+1} = k \\ j_1 + 2j_2 + \cdots + (n-k+1) j_{n-k+1} = n}} \frac{n!}{j_1! \cdots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}},$$

and the n^{th} complete exponential Bell polynomial is given by the sum

$$Y_n(x_1, \ldots, x_n) := \sum_{k=1}^{n} Y_{n,k}\left(x_1, \ldots, x_{n-k+1}\right) = \sum_{k = \left(\underbrace{1, \ldots, 1}_{k_1}, \ldots, \underbrace{n, \ldots, n}_{k_n}\right) \vdash n} \frac{n!}{k_1! \cdots k_n!} \left(\frac{x_1}{1!}\right)^{k_1} \cdots \left(\frac{x_n}{n!}\right)^{k_n}.$$

Cumulants

Theorem

$$B_n\left(\frac{1}{2}\right) = Y_n\left(0, -\frac{B_2}{2}, 0, \ldots, -\frac{B_n}{n}\right),$$

and

$$-\frac{B_{2n}}{2} = \sum_{k=1}^{2n} (-1)^{k-1}(k-1)!\,Y_{2n,k}\left(B_0\left(\frac{1}{2}\right), \ldots, B_{2n-k+1}\left(\frac{1}{2}\right)\right).$$

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The first result can be reduced to

$$Y_k\left(-\frac{B_2 \cdot 1!}{2 \cdot 2!}, \ldots, -\frac{B_{2k} \cdot k!}{(2k)\cdot(2k)!}\right) = \frac{k!\, B_{2k}\left(\frac{1}{2}\right)}{(2k)!} = \frac{k!}{(2k)!} \cdot \left(2^{2k-1} - 1\right) B_{2k}.$$

Theorem (M. Hoffman)

$$\frac{k!}{2^{2k}(2k+1)!} = Y_k\left(\frac{B_2 \cdot 1!}{2 \cdot 2!}, \frac{B_4 \cdot 2!}{4 \cdot 4!}, \ldots, \frac{B_{2k} \cdot k!}{(2k)\cdot(2k)!}\right).$$

Cumulants

Consider different moment generating function

$$M_Y(t) = \mathbb{E}\left[e^{tY}\right] = \frac{\sinh \frac{t}{2}}{\frac{t}{2}}$$

Theorem

It also holds that

$$\frac{B_{2n}}{2n} = \sum_{k=1}^{2n} (-1)^{k-1} (k-1)!\, Y_{2n,k}\left(0, \frac{1}{4 \cdot 3}, 0, \ldots, \frac{1+(-1)^{2n-k+2}}{2^{2n-k+2}(2n-k+2)}\right).$$

Cumulants

Consider different moment generating function

$$M_Y(t) = \mathbb{E}\left[e^{tY}\right] = \frac{\sinh \frac{t}{2}}{\frac{t}{2}}$$

Theorem

It also holds that

$$\frac{B_{2n}}{2n} = \sum_{k=1}^{2n} (-1)^{k-1} (k-1)!\, Y_{2n,k} \left(0, \frac{1}{4 \cdot 3}, 0, \ldots, \frac{1+(-1)^{2n-k+2}}{2^{2n-k+2}(2n-k+2)}\right).$$

$$Y_k \left(-\frac{B_2 \cdot 1!}{2 \cdot 2!}, \ldots, -\frac{B_{2k} \cdot k!}{(2k) \cdot (2k)!}\right) = \frac{k!}{(2k)!} \cdot \left(2^{2k-1} - 1\right) B_{2k}$$

$$Y_k \left(\frac{B_2 \cdot 1!}{2 \cdot 2!}, \frac{B_4 \cdot 2!}{4 \cdot 4!}, \ldots, \frac{B_{2k} \cdot k!}{(2k) \cdot (2k)!}\right) = \frac{k!}{2^{2k}(2k+1)!}$$

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$$f(x):=\sum_{k=0}^{\infty}\frac{B_{2k}k!}{(2k)\,(2k)!}=\log\left(\frac{e^x-1}{x}\right)-\frac{x}{2}$$

and

$$\sum_{k=0}^{\infty}\frac{x^{2k}}{2^{2k}\,(2k+1)!}=\frac{\sinh\left(\frac{x}{2}\right)}{\frac{x}{2}}=e^{f(x)}.$$

Continued Fractions & Orthogonal Polynomials

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Continued Fractions & Orthogonal Polynomials

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Continued Fractions & Orthogonal Polynomials

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Let $m_n = b_n = \left|B_n\left(\frac{1}{2}\right)\right|$, then $s_n = 0$ and $t_n = \frac{n^4}{4(2n+1)(2n-1)}$.

Hankel Determinant

$$\det\left(\left(m_{i+j}\right)_{i,j=0}^{n}\right) = \det\begin{pmatrix} m_0 & m_1 & \cdots & m_n \\ m_1 & m_2 & \cdots & m_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ m_n & m_{n+1} & \cdots & m_{2n} \end{pmatrix}.$$

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"Chapter 24"

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"Chapter 24" NIST:DLMF

$$\det\left(\left(B_{i+j}\right)_{i,j=0}^{n}\right) = (-1)^{\frac{n(n+1)}{2}} \frac{\left(\prod_{k=1}^{n} k!\right)^6}{\prod_{k=1}^{2n+1} k!}$$

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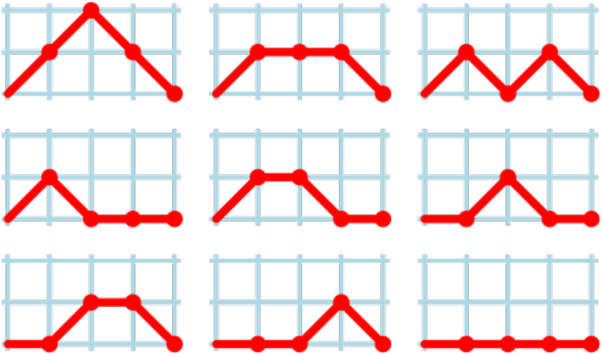
(Generalized) Motzkin Numbers



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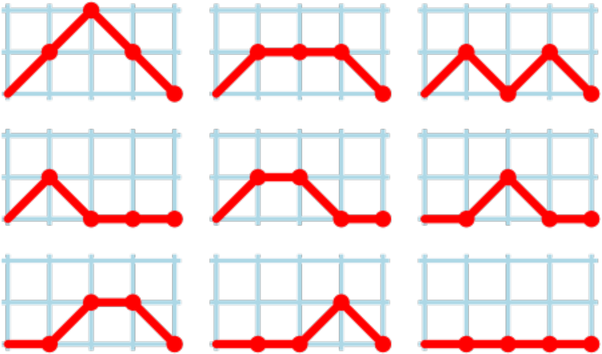

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Motzkin Numbers

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- If $s_k = 0$ and $t_k := -\frac{k(k+1)^2(k+2)}{4(2k+1)(2k+3)}$, then the corresponding $M_{n,0}$ gives $6B_{n+2}$:

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$$\Omega_{n+1}(x) = (2x+1)\,\Omega_n(x) - \frac{n^4}{(2n+1)(2n-1)}\Omega_{n-1}(x).$$

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Trick $\phi_n := 2^{-n}\Omega_n$. Moreover,

$$\mathcal{B}^r\Omega_n(\mathcal{B}) = \begin{cases} 0, & 0 \le r < n; \\ K_n, & r = n. \end{cases}$$

Motzkin Numbers

Recall the psi function

$$\psi(s) := \frac{\Gamma'(s)}{\Gamma(s)} = \frac{\mathrm{d}(\log\Gamma(s))}{\mathrm{d}s}$$

and

$$\psi'(s) = \frac{\mathrm{d}^2(\log\Gamma(s))}{\mathrm{d}s^2} = \sum_{k=0}^{\infty} \frac{1}{(k+s)^2} = \zeta(2,s).$$

On one hand, the asymptotic behavior

$$\psi'(x+1) \sim \sum_{k=0}^{\infty} \frac{B_k}{x^{k+1}},$$

while on the other hand, the continued fractions

$$\psi'(x) = \cfrac{a_1}{x - \frac{1}{2} + \cfrac{a_2}{x - \frac{1}{2} + \frac{a_3}{\cdots}}}, \text{ where } a_m = \begin{cases} 1, & m = 1; \\ \frac{(m-1)^4}{4(2m-3)(2m-1)}, & m \geq 2. \end{cases}$$

Or,

$$\psi'(x+1) = \cfrac{2}{2x+1+\cfrac{\lambda_1}{2x+1+\frac{\lambda_2}{\cdots}}}, \text{ where } \lambda_n = \frac{n^4}{4n^2-1}.$$

Then, define the polynomial sequence $(Q_n(x))_{n=0}^{\infty}$ by $Q_{-1} \equiv 0$, $Q_0 \equiv 1$ and

$$Q_{n+1}(x) = (2x+1)\,Q_n(x) + \lambda_n Q_{n-1}(x).$$

Motzkin Numbers

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$$\ln \Gamma(z) = \left(z - \frac{1}{2}\right) \ln z - z + \frac{\ln 2\pi}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} B_n}{2n(2n-1)} z^{-(2n-1)}$$

Motzkin Numbers

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$n = 0$:

$$\psi'(z+1) = \psi'(z) + \frac{1}{z} \sim \left(\frac{1}{z} + \frac{1}{2z^2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{z^{2k+1}}\right) + \frac{1}{z}$$

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$$M = \begin{pmatrix} \frac{1}{2} & 1 & 0 & 0 & 0 & \cdots \\ -\frac{1}{12} & \frac{1}{2} & 1 & 0 & 0 & \cdots \\ 0 & -\frac{4}{15} & \frac{1}{2} & 1 & 0 & \cdots \\ 0 & 0 & -\frac{81}{140} & \frac{1}{2} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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$$M_4 := \begin{pmatrix} \frac{1}{2} & 1 & 0 & 0 \\ -\frac{1}{12} & \frac{1}{2} & 1 & 0 \\ 0 & -\frac{4}{15} & \frac{1}{2} & 1 \\ 0 & 0 & -\frac{81}{140} & \frac{1}{2} \end{pmatrix} \Rightarrow M_4^4 = \begin{pmatrix} -\frac{1}{30} & -\frac{1}{5} & \frac{4}{7} & 2 \\ \frac{1}{60} & -\frac{13}{70} & -\frac{19}{14} & \frac{4}{7} \\ \frac{4}{315} & \frac{38}{105} & -\frac{689}{1470} & -\frac{25}{21} \\ -\frac{9}{350} & \frac{108}{1225} & \frac{135}{196} & -\frac{31}{98} \end{pmatrix}$$