The Probabilistic and Combinatorial Interpretations of the Bernoulli Symbol $\mathcal{B}$

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Outlines

Definition

The Bernoulli numbers $(B_n)_{n=0}^{\infty}$ and Bernoulli polynomials $(B_n(x))_{n=0}^{\infty}$ can be defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \text{ and } \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

with the relation

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k.$$

The Bernoulli symbol $\mathcal{B}$ satisfies the evaluation rule that

$$\mathcal{B}^n = B_n.$$

Treat $t = \partial_x$, and

$$\frac{t}{e^t - 1} \bullet x^n = B_n(x)$$

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{B}^{n-k} x^k = (\mathcal{B} + x)^n.$$

Examples

- $$B_n'(x) = nB_{n-1}(x) \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}x}(\mathcal{B}+x)^n = n(\mathcal{B}+x)^{n-1}.$$

- $$\begin{aligned} e^{\mathcal{B}t} = \frac{t}{e^t-1} \quad &\Rightarrow \quad e^{-\mathcal{B}t} = \frac{-t}{e^{-t}-1} = \frac{te^t}{e^t-1} = e^{(\mathcal{B}+1)t} \\ &\Rightarrow \quad -\mathcal{B} = \mathcal{B}+1 \\ &\Rightarrow \quad (-1)^n B_n(-x) = (-1)^n(\mathcal{B}-x)^n = B_n(x+1) \end{aligned}$$

Bernoulli (Random) Symbol

A. Dixit et al. show that let$L \sim \frac{\pi}{2}\operatorname{sech}^2(\pi t)$ and $\mathcal{B} \sim \imath L - \frac{1}{2}$, then

$$B_n = \mathbb{E}\left[\mathcal{B}^n\right] = \frac{\pi}{2}\int_{\mathbb{R}} \left(\imath t - \frac{1}{2}\right)^n \operatorname{sech}^2(\pi t)\, \mathrm{d}t$$

and

$$B_n(x) = \frac{\pi}{2}\int_{\mathbb{R}} \left(x + \imath t - \frac{1}{2}\right)^n \operatorname{sech}^2(\pi t)\, \mathrm{d}t = \mathbb{E}\left[(\mathcal{B} + x)^n\right].$$

By omitting expectation operator $\mathbb{E}$, we have

$$B_n = \mathcal{B}^n \text{ and } B_n(x) = (\mathcal{B} + x)^n.$$

Namely

$$\frac{t}{e^t - 1}\bullet = \mathbb{E}\left[\bullet\right]$$

Probabilistic Interpretation

For independent random variables X and Y, if $\mathbb{E}\left[e^{tX}\right] = F(x)$ and $\mathbb{E}\left[e^{tY}\right] = G(x)$, then

$$\mathbb{E}\left[e^{t(X+Y)}\right] = F(x)\, G(x)\, .$$

Choose $X = x$ and $Y = \mathcal{B}$, then

$$\mathbb{E}\left[e^{tX}\right] = e^{tx} \text{ and } \mathbb{E}\left[e^{t\mathcal{B}}\right] = \frac{t}{e^t - 1}$$

$$\mathbb{E}\left[e^{t(x+\mathcal{B})}\right] = \frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{\mathbb{E}\left[(x+\mathcal{B})^n\right]}{n!} t^n.$$

$$B_n(x) = \mathbb{E}\left[(\mathcal{B}+x)^n\right] = \frac{[t^n]\, e^{\mathcal{B}t} e^{xt}}{n!} = \frac{[t^n]\, \frac{te^{xt}}{e^t - 1}}{n!}.$$

Generalization

- Bernoulli:

$$\frac{t}{e^t - 1} e^{tx} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \Leftrightarrow B_n(x) = (x + \mathcal{B})^n$$

- Norlünd:

$$\left(\frac{t}{e^t - 1}\right)^p e^{tx} = \sum_{n=0}^{\infty} B_n^{(p)}(x) \frac{t^n}{n!} \Leftrightarrow B_n^{(p)}(x) = (x + \mathcal{B}_1 + \cdots + \mathcal{B}_p)^n$$

- Bernoulli-Barnes: for $\mathbf{a} = (a_1, \ldots, a_k)$.

$$e^{tx} \prod_{i=1}^{k} \frac{t}{e^{a_i t} - 1} = \sum_{n=0}^{\infty} B_n(\mathbf{a}; x) \frac{t^n}{n!}$$

$$\Leftrightarrow B_n(\mathbf{a}; x) = \frac{1}{|\mathbf{a}|} \left(x + \mathbf{a} \cdot \vec{\mathcal{B}}\right)^n, \text{ where } \begin{cases} \mathbf{a} = (a_1, \ldots, a_k) \\ \vec{\mathcal{B}} = (\mathcal{B}, \ldots, \mathcal{B}_k) \\ \mathbf{a} \cdot \vec{\mathcal{B}} = \sum_{l=1}^{k} a_l \mathcal{B}_l \\ |\mathbf{a}| = \prod_{l=1}^{k} a_l \end{cases}$$

Several Results

Bernoulli-Barnes

$$e^{tx}\prod_{i=1}^{k}\frac{t}{e^{a_i t}-1}=\sum_{n=0}^{\infty}B_n(\mathbf{a};x)\frac{t^n}{n!}\Leftrightarrow B_n(\mathbf{a};x)=\frac{1}{|\mathbf{a}|}\left(x+\mathbf{a}\cdot\vec{\mathcal{B}}\right)^n$$

Theorem[L. Jiu, V. Moll and C. Vignat]

$$f\left(x-\mathbf{a}\cdot\vec{\mathcal{B}}\right)=\sum_{\ell=0}^{n}\sum_{|L|=\ell}|\mathbf{a}|_{L^*}\,f^{(n-\ell)}\left(x+\left(\mathbf{a}\cdot\vec{\mathcal{B}}\right)_L\right).$$

The multiple zeta function

$$\zeta_r(n_1,\dots,n_r)=\sum_{0<k_1<\cdots<k_r}\frac{1}{k_1^{n_1}\cdots k_r^{n_r}}=\sum_{k_1,\dots,k_r=1}^{\infty}\frac{1}{k_1^{n_1}(k_1+k_2)^{n_2}\cdots(k_1+\cdots+k_r)^{n_r}}$$

Theorem[L. Jiu, V. Moll and C. Vignat]

$$\zeta_r(-n_1,\dots,-n_r)=\prod_{k=1}^{r}(-1)^{n_k}\,\mathcal{C}_{1,\dots,k}^{n_k+1},$$

where

$$\mathcal{C}_1^n=\frac{\mathcal{B}_1^n}{n},\ \mathcal{C}_{1,2}^n=\frac{(\mathcal{C}_1+\mathcal{B}_2)^n}{n},\dots,\mathcal{C}_{1,\dots,k+1}^n=\frac{\left(\mathcal{C}_{1,\dots,k}+\mathcal{B}_{k+1}\right)^n}{n}$$

Uniqueness of Hyperbolic Secant Square

$\mathcal{B} \sim \imath L - \frac{1}{2}$. Define $\bar{B}_n := \left|B_n\left(\frac{1}{2}\right)\right|$, then

$$\bar{B}_n = \frac{\pi}{2}\int_{\mathbb{R}} t^n \operatorname{sech}^2(\pi t)\,\mathrm{d}t. \left(\frac{x/2}{\sin(x/2)} = \sum_{n=0}^{\infty} \bar{B}_n \frac{x^n}{n!}\right)$$

Theorem

$\frac{\pi}{2}\operatorname{sech}^2(\pi t)\,\mathrm{d}t$ is the UNIQUE density on $\mathbb{R}$ for $\left(\bar{B}_n\right)_{n=0}^{\infty}$.

Proof.

$$(-1)^{n+1} B_{2n} \sim \frac{2(2n)!}{(2\pi)^{2n}}.$$

□

Lemma

Uniqueness is equivalent to existence of constants C and D, such that

$$\left|\bar{B}_n\right| \leq C D^n n!.$$

Cumulants

$$K(t) := \sum_{n=1}^{\infty} \kappa_n \frac{t^n}{n!} = \log \mathbb{E}\left[e^{tX}\right] = \log\left(\sum_{n=0}^{\infty} \frac{\mathbb{E}[X^n]}{n!} t^n\right).$$

Theorem

[Faà di Bruno's formula] For moments $(m_n)_{n=0}^{\infty}$ and cumulants $(\kappa_n)_{n=1}^{\infty}$ it holds that

$$m_n = Y_n(\kappa_1, \ldots, \kappa_n) \text{ and } \kappa_n = \sum_{k=1}^{n} (-1)^{k-1} (k-1)! Y_{n,k}(m_1, \ldots, m_{n-k+1}),$$

where, the partial or incomplete exponential Bell polynomial is given by

$$Y_{n,k}\left(x_1, \ldots, x_{n-k+1}\right) := \sum_{\substack{j_1+\cdots+j_{n-k+1}=k \\ j_1+2j_2+\cdots+(n-k+1)j_{n-k+1}=n}} \frac{n!}{j_1! \cdots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}},$$

and the n^{th} complete exponential Bell polynomial is given by the sum

$$Y_n(x_1, \ldots, x_n) := \sum_{k=1}^{n} Y_{n,k}\left(x_1, \ldots, x_{n-k+1}\right) = \sum_{k=\left(\underbrace{1, \ldots, 1}_{k_1}, \ldots, \underbrace{n, \ldots, n}_{k_n}\right) \vdash n} \frac{n!}{k_1! \cdots k_n!} \left(\frac{x_1}{1!}\right)^{k_1} \cdots \left(\frac{x_n}{n!}\right)^{k_n}.$$

Cumulants

Theorem

$$B_n\left(\frac{1}{2}\right) = Y_n\left(0, -\frac{B_2}{2}, -\frac{B_3}{3}, \ldots, -\frac{B_n}{n}\right),$$

and

$$B_n = -n \sum_{k=1}^{n} (-1)^{k-1} (k-1)! \, Y_{n,k}\left(B_0\left(\frac{1}{2}\right), \ldots, B_{n-k+1}\left(\frac{1}{2}\right)\right).$$

The first result can be reduced to

$$Y_k\left(-\frac{B_2 \cdot 1!}{2 \cdot 2!}, \ldots, -\frac{B_{2k} \cdot k!}{(2k) \cdot (2k)!}\right) = \frac{k! B_{2k}\left(\frac{1}{2}\right)}{(2k)!} = \frac{k!}{(2k)!} \cdot \left(2^{2k-1} - 1\right) B_{2k}.$$

Theorem (M. Hoffman)

$$Y_k\left(\frac{B_2 \cdot 1!}{2 \cdot 2!}, \frac{B_4 \cdot 2!}{4 \cdot 4!}, \ldots, \frac{B_{2k} \cdot k!}{(2k) \cdot (2k)!}\right) = \frac{k!}{2^{2k} (2k+1)!}.$$

Cumulants

Consider different moment generating function

$$M_Y(t) = \mathbb{E}\left[e^{tY}\right] = \frac{\sinh \frac{t}{2}}{\frac{t}{2}}$$

Theorem

It also holds that

$$B_n = n \sum_{k=1}^{n} (-1)^{k-1} (k-1)!\, Y_{n,k}\left(0, \frac{1}{4 \cdot 3}, 0, \ldots, \frac{1+(-1)^{n-k+2}}{2^{n-k+2}(n-k+2)}\right).$$

$$Y_k\left(-\frac{B_2 \cdot 1!}{2 \cdot 2!}, \ldots, -\frac{B_{2k} \cdot k!}{(2k) \cdot (2k)!}\right) = \frac{k!}{(2k)!} \cdot \left(2^{2k-1} - 1\right) B_{2k}$$

$$Y_k\left(\frac{B_2 \cdot 1!}{2 \cdot 2!}, \frac{B_4 \cdot 2!}{4 \cdot 4!}, \ldots, \frac{B_{2k} \cdot k!}{(2k) \cdot (2k)!}\right) = \frac{k!}{2^{2k}(2k+1)!}$$

Cumulants

$$Y_k\left(-\frac{B_2\cdot 1!}{2\cdot 2!},\ldots,-\frac{B_{2k}\cdot k!}{(2k)\cdot(2k)!}\right)=\frac{k!}{(2k)!}\cdot\left(2^{2k-1}-1\right)B_{2k}$$

$$Y_k\left(\frac{B_2\cdot 1!}{2\cdot 2!},\frac{B_4\cdot 2!}{4\cdot 4!},\ldots,\frac{B_{2k}\cdot k!}{(2k)\cdot(2k)!}\right)=\frac{k!}{2^{2k}\left(2k+1\right)!}$$

$$f\left(x\right):=\sum_{k=0}^{\infty}\frac{B_{2k}k!}{\left(2k\right)\left(2k\right)!}=\log\left(\frac{e^x-1}{x}\right)-\frac{x}{2}$$

and

$$\sum_{k=0}^{\infty}\frac{x^{2k}}{2^{2k}\left(2k+1\right)!}=\frac{\sinh\left(\frac{x}{2}\right)}{\frac{x}{2}}=e^{f(x)}.$$

Continued Fractions & Orthogonal Polynomials

$$(m_n)_{n=0}^{\infty} \sim m_n = \int_{\mathbb{R}} x^n d\mu(x) \quad \overset{?}{\Rightarrow} \quad (P_n(x))_{n=1}^{\infty} \sim \int_{\mathbb{R}} P_n(x) P_m(x) \,\mathrm{d}\mu(x) = C_n \delta_{m,n}$$

$$\Rightarrow \quad P_{n+1}(x) = (x + s_n) P_n(x) - t_n P_{n-1}(x)$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} m_n x^n = \cfrac{m_0}{1 - s_0 x - \cfrac{t_1 x^2}{1 - s_1 x - \cfrac{t_2 x^2}{1 - \cdots}}}$$

Theorem [J. Touchard]

The polynomial sequence (ϕ_n), define by

$$\phi_{n+1}(z) = \left(z + \frac{1}{2}\right) \phi_n(z) + \omega_n \phi_{n-1}(z)$$

satisfies for any $0 \leq r < n$, $\underline{\mathcal{B}^r \phi_n(\mathcal{B}) = 0}$, where

$$\omega_n = \frac{n^4}{4(2n+1)(2n-1)}.$$

How

$$\psi_1(z) := \psi'(z) := (\log(\Gamma(z)))''$$

$$\sum_{n=0}^{\infty} \frac{B_n}{z^{n+1}} \sim \psi_1(z+1) = \cfrac{1}{z+\frac{1}{2}+\cfrac{\omega_1}{z+\frac{1}{2}+\cfrac{\omega_2}{z+\frac{1}{2}+\cdots}}}$$

$$\psi(z+x) \sim \log(z) - \sum_{n=1}^{\infty} \frac{(-1)^n B_n(x)}{nz^n}$$

A. Dixie et al. showed

$$\log(\mathcal{B}+z) = \psi\left(\left|z-\frac{1}{2}\right|+\frac{1}{2}\right).$$

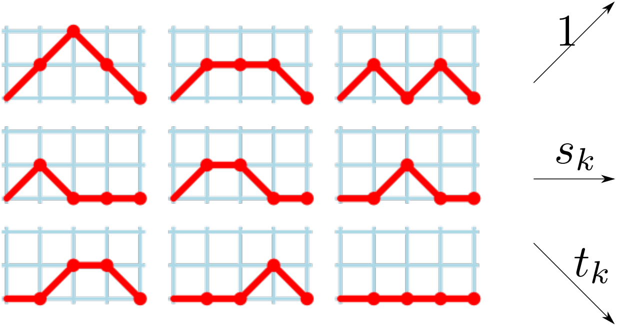
Theorem

$$\varphi_{n+1}(z,x) := \left(z+\frac{1}{2}-x\right)\varphi_n(z,x) + \omega_n\varphi_{n-1}(z,x)$$

$$z^r\varphi_n(z,x)\Big|_{z=\mathcal{B}+r} = (\mathcal{B}+x)^r\varphi_n(\mathcal{B}+x,x) = 0,\ \forall 0 \le r < n.$$

(Generalized) Motzkin Numbers

$$M_{n+1,k} = M_{n,k-1} + s_k M_{n,k} + t_{k+1} M_{n,k+1}$$



$$\sum_{n=0}^{\infty} M_{n,0} z^n = \cfrac{1}{1 - s_0 z - \cfrac{t_1 z^2}{1 - s_1 z - \cfrac{t_2 z^2}{\cdots}}}$$

Combinatorial Interpretation

Theorem

Define $\left(M_{n,k}^{\mathbf{x},\omega}\right)_{n,k=0}^{\infty}$, *by* $M_{0,0}^{\mathbf{x},\omega}=1$, $M_{n,k}^{\mathbf{x},\omega}=0$ if $k>n$, *and the recurrence*

$$M_{n+1,k}^{\mathbf{x},\omega}=M_{n,k-1}^{\mathbf{x},\omega}+x_k M_{n,k}^{\mathbf{x},\omega}-\omega_{k+1}M_{n,k+1}^{\mathbf{x},\omega},$$

where $\mathbf{x}=(x_n)_{n=0}^{\infty}$ *is given by* $x_n=x-\frac{1}{2}$, *and* $\omega=(\omega_n)_{n=1}^{\infty}$ *by* $\omega_n=\frac{n^4}{4(2n+1)(2n-1)}$. *Then,* $M_{n,0}^{\mathbf{x},\omega}=B_n(x)$. *In addition, the lattice path interpretation allows us to define the infinite-dimensional matrix*

$$R_{\mathbf{x},\omega}:=\begin{pmatrix} x-\frac{1}{2} & -\omega_1 & 0 & 0 & \cdots & 0 & \cdots \\ 1 & x-\frac{1}{2} & -\omega_2 & 0 & \cdots & 0 & \cdots \\ 0 & 1 & x-\frac{1}{2} & \ddots & \ddots & \vdots & \cdots \\ 0 & 0 & 1 & \ddots & -\omega_n & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & x-\frac{1}{2} & -\omega_{n+1} & \cdots \\ 0 & 0 & 0 & \ddots & 1 & \ddots & \ddots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

Matrix Computation

Direct computations shows

$$R_{\mathbf{x},\omega,4} = \begin{pmatrix} x-1/2 & -\frac{1}{12} & 0 & 0 \\ 1 & x-1/2 & -\frac{4}{15} & 0 \\ 0 & 1 & x-1/2 & -\frac{81}{140} \\ 0 & 0 & 1 & x-1/2 \end{pmatrix}$$

and

$$\left(R^4_{\mathbf{x},\omega,4}\right) = \begin{pmatrix} x^4-2x^3+x^2-\frac{1}{30} & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix},$$

where noting

$$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}.$$

Euler Analogue

Definition

Euler numbers $(E_n)_{n=0}^{\infty}$ and Euler polynomials $(E_n(x))_{n=0}^{\infty}$

$$\operatorname{sech}(t) = \frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \quad \text{and} \quad \frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

In addition,

$$E_n(x) = \int_{\mathbb{R}} \left(x - \frac{1}{2} + \imath t\right)^n \operatorname{sech}(\pi t)\, \mathrm{d}t.$$

$E_n = 2^n E_n\left(\frac{1}{2}\right) \Rightarrow \mathcal{E} \sim 2\imath L_E$, where L_E has its density function $\operatorname{sech}(\pi t)$.

$$\mathcal{E}^n := \mathbb{E}\left[\mathcal{E}^n\right] = E_n$$

Conversely, it holds that $\mathbb{E}\left[L_E^n\right] = \left(\frac{\imath}{2}\right)^n E_n$ and $\mathbb{E}\left[e^{tL_E}\right] = \sec\left(\frac{t}{2}\right)$.

Euler Analogue

- Uniqueness of sech(πt) for L_E ✓ $(-1)^n E_{2n} \sim 8\sqrt{n/\pi}\,(4n/\pi/e)^{2n}$
- Faà di Bruno's formula:

$$\begin{cases} E_{2n} = 1 - \sum_{k=1}^{n} \binom{2n}{2k-1} \frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{2k} \\ B_{2n} = \frac{2n}{2^{2n}(2^{2n}-1)} \sum_{k=0}^{n-1} \binom{2n-1}{2k} E_{2k} \end{cases}$$

- Orthogonal polynomials, Motzkin number, continued fractions

$$2\beta\left(\frac{s+1}{2}\right) \sim \sum_{j=1}^{\infty} \frac{E_j}{s^{j+1}}$$

Possible Extension to Nörlund Polynomials

$$\left(\frac{t}{e^t-1}\right)^p e^{xt} = \sum_{n=0}^{\infty} B_n^{(p)}(x)\frac{t^n}{n!} \Leftrightarrow B_n^{(p)}(x) = \left(\mathcal{B}_1+\cdots+\mathcal{B}_p+x\right)^n.$$

$$\frac{\Gamma(z+x)}{\Gamma(z+x+1-p)\,z^p} \sim \sum_{n=0}^{\infty}\frac{(p-n)_n}{n!}B_n^{(p)}(x)\frac{1}{z^{n+1}}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$.

$$\log\left(\mathcal{B}_1+\cdots+\mathcal{B}_p+z\right) = -H_{p-1}+\frac{\mathrm{d}^{p-1}}{\mathrm{d}z^{p-1}}\left[\binom{z-1}{p-1}\psi\left(z-\lfloor\frac{p}{2}\rfloor\right)\right]$$

where $H_n := 1+1/2+\cdots+1/n$, is the n-th harmonic number and $\lfloor\rfloor$ is the floor function.

End

Thank you