Two Sequences Related to Bernoulli and Euler Numbers

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May 30th, 2018

Two Sequences

- $$1, 0, -\frac{1}{12}, 0, \frac{7}{240}, 0, -\frac{31}{1344}, 0$$

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Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \qquad \frac{2e^{zx}}{e^z + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!}$$

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n	0	1	2	3	4
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$
$B_n(x)$	1	$x - \frac{1}{2}$	$x^2 - x + \frac{1}{6}$	$x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$	$x^4 - 2x^3 + x^2 - \frac{1}{30}$
E_n	1	0	-1	0	5
$E_n(x)$	1	$x - \frac{1}{2}$	$x^2 - x$	$x^3 - \frac{3}{2}x^2 + \frac{1}{4}$	$x^4 - 2x^3 + x$

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n	0	1	2	3	4	5	6	7	8
$B_n\left(\frac{1}{2}\right)$	1	0	$-\frac{1}{12}$	0	$\frac{7}{240}$	0	$-\frac{31}{1344}$	0	$\frac{127}{3840}$

Guessing Formulas

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$$1, 2, 3,$$

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$$1, 2, 3, 5, 7$$

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$$1, 2, 3, 5, 7, 11,$$

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$$1, 2, 3, 5, 7, 11, 15, 22, \cdots$$

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Parition of numbers

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- $4 = 4$
- $4 = 3 + 1$
- $4 = 2 + 2$
- $4 = 2 + 1 + 1$
- $4 = 1 + 1 + 1 + 1$

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Theorem. Given polynomial $P(x)$ with $\deg P = d$, we have

$$\sum_{k=1}^{n} P(k) = Q(n),$$

for some polynomials $Q(x)$ with $\deg Q = d + 1$.

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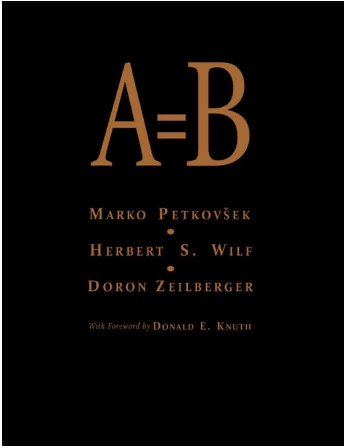

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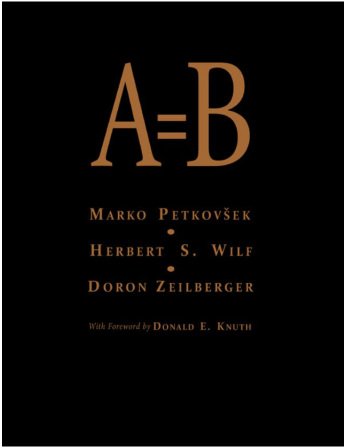

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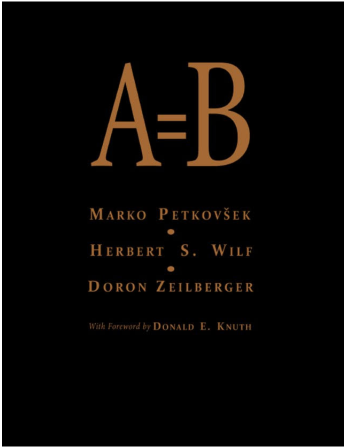

- $$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$
- $$\sum_{k=0}^{\infty} (-1)^k \binom{n}{k} \binom{2k}{k} 4^{n-k} = \binom{2n}{n}$$

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In particular,

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- Moreover, P_n satisfies a three-term recurrence: for $n > 1$,

$$P_{n+1}(y) = (y + s_n)P_n(y) + t_n P_{n-1}(y).$$

Bernoulli

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Proof. By induction on the degree of P.

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Lemma. [L. Jiu and D. Shi]

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Proposition. Let $\phi_n = \sum \alpha_{n,k} y^{n-2k}$, then

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[Question2] Find the closed form for ϕ_n.

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[Question2] Find the closed form for ϕ_n.

$$\tau_0 = [\lambda_n, \cdots, \lambda_0]\left\{\text{ last column of } \left(A_n^{-1}\right)\right\}, A_n = \begin{bmatrix} \alpha_{n,n} & 0 & \alpha_{n,n-1} & \cdots & \\ 0 & \alpha_{n-1,n-1} & 0 & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Last Column: $1, 0, -\frac{1}{12}, 0, \frac{7}{240}, 0, -\frac{31}{1344}, 0, \frac{16473530237}{53003808000}, \ldots$

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$$\begin{bmatrix} \varphi_4 \\ \varphi_3 \\ \varphi_2 \\ \varphi_1 \\ \varphi_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 14 & 0 & 9 \\ 0 & 1 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y^4 \\ y^3 \\ y^2 \\ y \\ 1 \end{bmatrix}$$

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Example.

$$\varphi_n(y) = i^n n! P_n^{(\frac{1}{2})}\left(\frac{-iy}{2}; \frac{\pi}{2}\right)$$

End

End and Sage