Matrix Representations for Bernoulli and Euler Polynomials

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Objects

Bernoulli polynomials:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

Euler polynomials:

$$\frac{2e^{zx}}{e^z + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!}.$$

Bernoulli numbers: $B_n = B_n(0)$

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \qquad B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k.$$

n	0	1	2	3	4
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$
$B_n(x)$	1	$x - \frac{1}{2}$	$x^2 - x + \frac{1}{6}$	$x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$	$x^4 - 2x^3 + x^2 - \frac{1}{30}$

$$B_n^+ := \begin{cases} B_n, & \text{if } n \neq 1; \\ 1/2, & \text{if } n = 1. \end{cases} \qquad B_n^+ = B_n(1).$$

Matrix & Determinants

$$B_n^+ = \frac{\det\left(a_{i,j}\right)_{n\times n}}{n!}, \quad a_{i,j} := \begin{cases} 0, & \text{if } j > i+1; \\ \binom{i+1}{j-1}, & \text{otherwise.} \end{cases}$$

$$B_3 = B_3^+ = \frac{\det\begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & 3 \\ 1 & 4 & 6 \end{pmatrix}}{3!} = 0.$$

$$B_n = (-1)^n n! \det\begin{pmatrix} \frac{1}{2!} & 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & \cdots & 0 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \frac{1}{(n-3)!} & \cdots & 1 \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \cdots & \frac{1}{2!} \end{pmatrix}$$

Matrix Representation

$$R := \begin{pmatrix} x-\frac{1}{2} & \omega_1 & 0 & 0 & \cdots & \cdots \\ 1 & x-\frac{1}{2} & \omega_2 & 0 & \cdots & \cdots \\ 0 & 1 & x-\frac{1}{2} & \ddots & \ddots & \cdots \\ 0 & 0 & 1 & \ddots & \omega_n & \cdots \\ 0 & 0 & 0 & \ddots & x-\frac{1}{2} & \cdots \\ 0 & 0 & 0 & \ddots & 1 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \text{where } \omega_n = -\frac{n^4}{4(2n+1)(2n-1)}.$$

For exmaple,

$$R_4 := \begin{pmatrix} x-1/2 & -1/12 & 0 & 0 \\ 1 & x-1/2 & -4/15 & 0 \\ 0 & 1 & x-1/2 & -81/140 \\ 0 & 0 & 1 & x-1/2 \end{pmatrix}$$

$$(R_4)^4 = \begin{pmatrix} x^4-2x^3+x^2-\frac{1}{30} & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

If letting $\omega_n = -n^2/4$, we could similarly obain $E_n(x)$.

Main Result

Theorem [L. J and D. Shi]

Define $M_{n,k}$ by $M_{0,0} = 1$ and for $n > 0$

$$M_{n+1,k} = M_{n,k-1} + \left(x - \frac{1}{2}\right) M_{n,k} + \omega_{k+1} M_{n,k+1},$$

where $\omega_k = -\frac{k^4}{4(2k+1)(2k-1)}$ and $M_{n,k} = 0$ if $k > n$. Then, $M_{n,0} = B_n(x)$.

$$R := \begin{pmatrix} x-\frac{1}{2} & \omega_1 & 0 & 0 & \cdots & \cdots \\ 1 & x-\frac{1}{2} & \omega_2 & 0 & \cdots & \cdots \\ 0 & 1 & x-\frac{1}{2} & \ddots & \ddots & \cdots \\ 0 & 0 & 1 & \ddots & \omega_n & \cdots \\ 0 & 0 & 0 & \ddots & x-\frac{1}{2} & \cdots \\ 0 & 0 & 0 & \ddots & 1 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Random Variable

Let X be a random variable with density function $p(t)$ on $\mathbb{R}$ and with moments m_n, i.e.,

$$m_n = \mathbb{E}[X^n] = \int_{\mathbb{R}} t^n p(t)\mathrm{d}t.$$

Let $P_n(y)$ be the monic orthogonal polynomials with respect to X (or w. r. t. m_n), i.e., $\deg P_n = n$, $\mathrm{LC}[P_n] = 1$, and

$$\int_{\mathbb{R}} P_m(t)P_n(t)p(t)\mathrm{d}t = c_n\delta_{m,n} = \begin{cases} c_n, & \text{if } m = n; \\ 0, & \text{otherwise.} \end{cases}$$

$$c_n\delta_{m,n} = P_m(t)P_n(t)\Big|_{t^k = m_k}.$$

P_n satisfies a three-term recurrence: for some sequences $(s_n)_{n\geq 0}$ and $(t_n)_{n\geq 1}$,

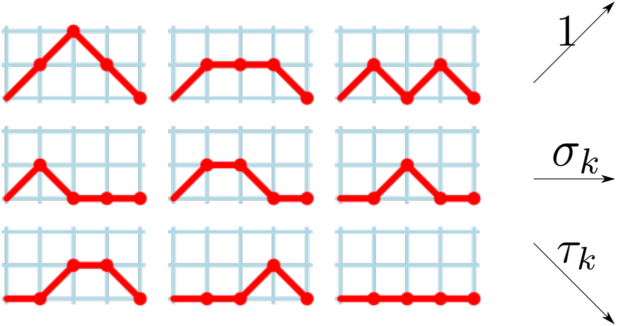
$$P_{n+1}(y) = (y - s_n)P_n(y) - t_n P_{n-1}(y).$$

Theorem

$$\sum_{n=0}^{\infty} m_n x^n = \frac{m_0}{1 - s_0 x - \frac{t_1 x^2}{1 - s_1 x - \frac{t_2 x^2}{1 - \cdots}}}.$$

Generalized Motzkin Numbers

$$M_{n+1,k} = M_{n,k-1} + \sigma_k M_{n,k} + \tau_{k+1} M_{n,k+1}$$



$$\sum_{n=0}^{\infty} M_{n,0} z^n = \cfrac{1}{1 - s_0 z - \cfrac{t_1 z^2}{1 - s_1 z - \cfrac{t_2 z^2}{\cdots}}}$$

General Identification

Theorem

For random variable X with moments m_n and monic orthogonal polynomials P_n, satisfying recurrence

$$P_{n+1}(y) = (y - s_n)P_n(y) - t_n P_{n-1}(y),$$

we define generalized Motzkin numbers $M_{n,k}$ by letting $(\sigma_k, \tau_k) = (s_k, t_k)$. If further assuming $m_0 = 1$, we have

$$M_{n,0} = m_n = \mathbb{E}[X^n].$$

$$M_{n+1,k} = M_{n,k-1} + s_k M_{n,k} + t_{k+1} M_{n,k+1}$$

$$M_{n+1,0} = s_0 M_{n,0} + t_1 M_{n,1}.$$

Probabilistic Interpretation of $B_n(x)$

$$M_{n+1,k} = M_{n,k-1} + \left(x - \frac{1}{2}\right) M_{n,k} + \omega_{k+1} M_{n,k+1} \Rightarrow M_{n,0} = B_n(x).$$

[Question1] Are $B_n(x)$ moments of some random variable?
[Answer] Let $L_B \sim \pi \operatorname{sech}^2(\pi t)/2$, then

$$B_n(x) = \frac{\pi}{2} \int_{\mathbb{R}} \left(x + it - \frac{1}{2}\right)^n \operatorname{sech}^2(\pi t) \, \mathrm{d}t = \mathbb{E}\left[\left(iL_B + x - \frac{1}{2}\right)^n\right].$$

[Question2] What are the monic orthogonal polynomials w. r. t. $B_n(x)$?
[Answer] Well...

- 1st oberservation: $B_n = \mathbb{E}\left[\left(iL_B - \frac{1}{2}\right)^n\right] = B_n(0)$.

One More Setup

Theorem (J. Touchard)

Let ϕ_n be the monic orthogonal polynomials w. r. t. B_n. Then, they satisfy

$$\phi_{n+1}(y) = \left(y + \frac{1}{2}\right)\phi_n(y) - \omega_n \phi_{n-1}(y) \quad \left[(s_n, t_n) = \left(\frac{1}{2}, \omega_n\right)\right].$$

	Moments	Monic Orthogonal Polynomials
X	m_n	$P_{n+1}(y) = (y - s_n)P_n(y) - t_n P_{n-1}(y)$
$X + c$	$\sum_{k=0}^{n} \binom{n}{k} m_k c^{n-k}$	$Q_{n+1}(y) = (y - s_n - c)Q_n(y) - t_n Q_{n-1}(y)$
CX	$C^n m_n$	$R_{n+1}(y) = (y - Cs_n)R_n(y) - C^2 t_n R_{n-1}(y)$

Recall

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_n x^{n-k}.$$

Conclusion

For Bernoulli polynomials $B_n(x)$, the corresponding orthgonal polynomials $\theta_n(y)$ satisfy

$$\theta_{n+1}(y) = \left(y - \left(x - \frac{1}{2}\right)\right)\theta_n(y) - \omega_n\theta_{n-1}(y).$$

Thus, we define the generalized Motzkin numbers by

$$M_{n+1,k} = M_{n,k-1} + \left(x - \frac{1}{2}\right)M_{n,k} + \omega_{k+1}M_{n,k+1},$$

which implies $M_{n,0} = B_n(x)$. The matrix presentation follows from the lattice path interpretation.

Euler Polynoimials

Euler numbers $(E_n)_{n=0}^{\infty}$ and Euler polynomials $(E_n(x))_{n=0}^{\infty}$

$$\operatorname{sech}(t) = \frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \quad \text{and} \quad \frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

In addition,

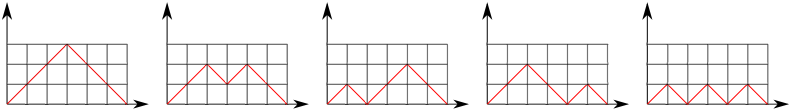
$$E_n(x) = \int_{\mathbb{R}} \left(x - \frac{1}{2} + it\right)^n \operatorname{sech}(\pi t)\, \mathrm{d}t.$$

$$E_n = 2^n E_n\left(\frac{1}{2}\right)$$

$$M_{n+1,k} = M_{n,k-1} + \left(x - \frac{1}{2}\right) M_{n,k} - \frac{(k+1)^2}{4} M_{n,k+1}$$

Euler Polynoimials

$$E_n = 2^n E_n\left(\frac{1}{2}\right) \Rightarrow M_{n+1,k} = M_{n,k-1} - \frac{(k+1)^2}{4} M_{n,k+1}$$



$n = 6,\ E_6$ Dyck paths

$$C_3 = \frac{1}{4}\binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{4 \cdot 3 \cdot 2 \cdot 1} = 5.$$

$$-61 = E_6 = (-1)^3 \left(3^2 2^2 1^2 + 2^2 2^2 1^2 + 1^2 2^2 1^2 + 2^2 1^2 1^2 + 1^2 1^2 1^2\right)$$

Currenct Work

- Dyck paths$\Rightarrow(E_{2n}, C_n)$;
- Nörlund polynomials :

$$\left(\frac{t}{e^t-1}\right)^p e^{xt} = \sum_{n=0}^{\infty} B_n^{(p)}(x)\frac{t^n}{n!};$$

$$(R_4)^4$$

$$\left(\begin{array}{rrr}
x^4-2x^3+x^2-\frac{1}{30} & -\frac{1}{3}x^3+\frac{1}{2}x^2-\frac{2}{15}x-\frac{1}{135} & \frac{2}{15}x^2 \\
4x^3-6x^2+\frac{8}{5}x+\frac{4}{45} & x^4-2x^3-\frac{3}{5}x^2+\frac{52}{45}x-\frac{37}{3780} & -\frac{16}{15}x^3+\frac{14}{15}x^2 \\
6x^2-\frac{13}{3}x-\frac{89}{1008} & 4x^3-\frac{7}{2}x^2-\frac{635}{252}x+\frac{1673}{1728} & x^4-\frac{1}{3}x^3-\frac{845}{168}x^2+ \\
4x-\frac{19}{12} & 6x^2-\frac{13}{3}x-\frac{89}{1008} & 4x^3-\frac{7}{2}x^2
\end{array}\right.$$

End

Thank you