Bernoulli Symbol and Sum of Powers

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Joint Work

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Bernoulli Numbers & Bernoulli Polynomials

Bernoulli numbers B_n and Bernoulli polynomials $B_n(x)$

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \text{ and } \frac{t e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

Examples

$$1^n + 2^n + \cdots + N^n = \frac{1}{n+1} \sum_{i=1}^{n+1} \binom{n+1}{i} B_{n+1-i} N^i = \frac{B_{n+1}(N+1) - B_{n+1}}{n+1}.$$

Riemann-zeta: for $n \in \mathbb{Z}_+$

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n} B_{2n}}{2(2n)!}, \ \zeta(-n) = -\frac{B_{n+1}}{n+1} = (-1)^n \frac{B_{n+1}}{n+1}.$$

$$B_{2n+1} = 0$$

Bernoulli Symbol (Umbral)

Key Idea:

$$\mathcal{B}^n \mapsto B_n : \text{ i.e., super index}\leftrightarrow\text{lower index.}$$

Why?

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k = (\mathcal{B} + x)^n .$$

$$B_n'(x) = nB_{n-1}(x)$$

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$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k = (\mathcal{B} + x)^n .$$

$$B_n'(x) = nB_{n-1}(x) \Leftrightarrow [(\mathcal{B} + x)^n]' = n(\mathcal{B} + x)^{n-1} .$$

Probabilitistic Interpretation

Let $L_B \sim \frac{\pi}{2}\operatorname{sech}^2(\pi t)$, then $\mathcal{B} \sim iL_B - \frac{1}{2}$

$$B_n = \mathcal{B}^n = \mathbb{E}[\mathcal{B}^n] = \frac{\pi}{2}\int_{\mathbb{R}} \left(it - \frac{1}{2}\right)^n \operatorname{sech}^2(\pi t)\, dt.$$

$$B_n(x) = (\mathcal{B} + x)^n = \frac{\pi}{2}\int_{\mathbb{R}} \left(x + it - \frac{1}{2}\right)^n \operatorname{sech}^2(\pi t)\, dt.$$

Probabilitistic Interpretation

- $$B_n = \mathcal{B}^n = \mathbb{E}\left[\mathcal{B}^n\right] = \frac{\pi}{2}\int_{\mathbb{R}}\left(it - \frac{1}{2}\right)^n \operatorname{sech}^2(\pi t)\, dt.$$

- $$B_n(x) = (\mathcal{B} + x)^n = \frac{\pi}{2}\int_{\mathbb{R}}\left(x + it - \frac{1}{2}\right)^n \operatorname{sech}^2(\pi t)\, dt. \left(\frac{t}{e^t - 1} \mid e^{tx}\right)$$

- Norlünd:

$$\left(\frac{t}{e^t - 1}\right)^p e^{tx} = \sum_{n=0}^{\infty} B_n^{(p)}(x)\frac{t^n}{n!} \Leftrightarrow B_n^{(p)} = \Big(\underbrace{\mathcal{B}_1 + \cdots + \mathcal{B}_p}_{\text{i. i. d.}} + x\Big)^n$$

- Bernoulli-Barnes: for $\mathbf{a} = (a_1, \ldots, a_p)$, $|\mathbf{a}| = \prod_{l=1}^{p} a_l \neq 0$, and $\vec{\mathcal{B}} = (\mathcal{B}_1, \ldots, \mathcal{B}_p)$,

$$e^{tx}\prod_{i=1}^{p}\frac{t}{e^{a_i t} - 1} = \sum_{n=0}^{\infty} B_n(\mathbf{a}; x)\frac{t^n}{n!} \Leftrightarrow B_n(\mathbf{a}; x) = \frac{1}{|\mathbf{a}|}\left(x + \mathbf{a}\cdot\vec{\mathcal{B}}\right)^n$$

Multiple Zeta Values

Riemann-zeta: for $n \in \mathbb{Z}_{+}$, the AC $\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$.

DEF.

$$\zeta_r(n_1, \ldots, n_r) = \sum_{0<k_1<\cdots<k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}} = \sum_{k_1, \ldots, k_r = 0}^{\infty} \frac{1}{k_1^{n_1} (k_1 + k_2)^{n_2} \cdots (k_1 + \cdots + k_r)^{n_r}}$$

Theorem(L. Jiu, V. H. Moll and C. Vignat)

$$\zeta_r(-n_1, \ldots, -n_r) = \prod_{k=1}^{r} (-1)^{n_k} \mathcal{C}_{1,\ldots,k}^{n_k+1},$$

where

$$\mathcal{C}_1^n = \frac{\mathcal{B}_1^n}{n}, \mathcal{C}_{1,2}^n = \frac{(\mathcal{C}_1 + \mathcal{B}_2)^n}{n}, \ldots, \mathcal{C}_{1,\ldots,k+1}^n = \frac{(\mathcal{C}_{1,\ldots,k} + \mathcal{B}_{k+1})^n}{n}$$

Example

$$\begin{aligned} \zeta_2(-n, 0) &= (-1)^n \mathcal{C}_1^{n+1} \cdot (-1)^0 \mathcal{C}_{1,2}^{0+1} \\ &= (-1)^n \frac{\mathcal{C}_1 + \mathcal{B}_2}{1} \cdot \mathcal{C}_1^{n+1} \\ &= (-1)^n \left(\mathcal{C}_1^{n+2} + \mathcal{B}_2 \mathcal{C}_1^{n+1}\right) \\ &= (-1)^n \left[\frac{B_{n+2}}{n+2} - \frac{1}{2} \frac{B_{n+1}}{n+1}\right]. \end{aligned}$$

$$\zeta(-n) = (-1)^n \mathcal{C}^{n+1} = (-1)^n \frac{B_{n+1}}{n+1}.$$

Two different ACs.

Analytic Continuation: for $n_1, \ldots, n_r$ positive integers

Theorem(Sadaoui)

$$\zeta_r(-n_1, \ldots, -n_r) = (-1)^r \sum_{k_2,\ldots,k_r} \frac{1}{\bar{n}+r-\bar{k}} \prod_{j=2}^{r} \frac{\binom{\sum_{i=j}^{r} n_i - \sum_{i=j+1}^{n} k_i + r - j + 1}{k_j}}{\sum_{i=j}^{r} n_i - \sum_{i=j}^{n} k_i + r - j + 1}$$

$$\times \sum_{l_1,\ldots,l_r} \binom{\bar{n}+r-\bar{k}}{l_1} \binom{k_2}{l_2} \cdots \binom{k_r}{l_r} B_{l_1} \cdots B_{l_r},$$

$\bar{n} = \sum_{j=1}^{n} n_j$, $\bar{k} = \sum_{j=2}^{r} k_j$, $k_2, \ldots k_r \geq 0$, $l_j \leq k_j$ for $2 \leq j \leq r$ and $l_1 \leq \bar{n} + r + \bar{k}$.

Theorem(Akiyama and Tanigawa)

$$\begin{aligned}\zeta_r(-n_1, \ldots, -n_r) = & -\frac{\zeta_{r-1}(-n_1, \ldots, -n_{r-2}, -n_{r-1} - n_r - 1)}{1+n_r} \\ & -\frac{\zeta_{r-1}(-n_1, \ldots, -n_{r-2}, -n_{r-1} - n_r)}{2} \\ & + \sum_{q=1}^{n_r} (-n_r)_q \, a_q \zeta_{r-1}(-n_1, \ldots, -n_{r-2}, -n_{r-1} - n_r + q),\end{aligned}$$

Sum of Powers

$$\sum_{N>i_1>\cdots>i_r>0} \frac{1}{i_1^{n_1}\cdots i_r^{n_r}} \left(\sum_{N>i_1>\cdots>i_r>0} i_1^{n_1}\cdots i_r^{n_r} \right)$$

Faulhaber's formula

$$\sum_{k=1}^{N} k^n = \frac{B_{n+1}(N+1) - B_{n+1}}{n+1}, \; n \geq 1, \; N \geq 1$$

G. H. E. Duchamp, V. H. N. Minh and N. Q. Hoan, Harmonic sums and polylogarithms at non-positive multi-indices, *J. Symbolic Comput.* **83** (2017), 166–186.

$\mathcal{H}$ Symbol

$$(\mathcal{H}(N))^n = H_{-n}(N) = 1^n + 2^n + \cdots + (N-1)^n = \sum_{N>k>0} k^n.$$

Faulhaber's formula

$$(\mathcal{H}(N))^n = \sum_{k=1}^{N-1} k^n = \frac{B_{n+1}(N) - B_{n+1}}{n+1} = \int_0^N (\mathcal{B} + x)^n \, \mathrm{d}x$$

$y \in \mathbb{C}$

$$(\mathcal{H}(y))^n = \frac{1}{n+1} \sum_{k=1}^{n+1} \binom{n+1}{k} B_{n+1-k} y^k$$

$$H_{-n_1,\ldots,-n_r}(N) = \sum_{N>i_1>\cdots>i_r>0} i_1^{n_1} \cdots i_r^{n_r}.$$

Sum of Powers

Theorem(L. J and C. Vignat)

$$H_{-n_1,\ldots,-n_r}(N) = \prod_{k=1}^{r} \mathcal{H}_{1,\ldots,k}^{n_k},$$

where $\mathcal{H}_1 = \mathcal{H}(N)$ and recursively $\mathcal{H}_{1,\ldots k} = \mathcal{H}(\mathcal{H}_{1,\ldots,k-1})$. $r = 2$:

$$\begin{aligned}
H_{-n,-m}(N) &= \sum_{N>i>j>0} i^n j^m = \mathcal{H}_1^n \cdot \mathcal{H}_{1,2}^m = \mathcal{H}_1^n \left[\mathcal{H}(\mathcal{H}_1)\right]^m \\
&= \frac{\mathcal{H}_1^n}{m+1} \sum_{k=1}^{m+1} \binom{m+1}{k} B_{m+1-k} \mathcal{H}_1^k = \frac{1}{m+1} \sum_{k=1}^{m+1} \binom{m+1}{k} B_{m+1-k} \mathcal{H}_1^{n+k} \\
&= \frac{1}{m+1} \sum_{k=1}^{m+1} \binom{m+1}{k} B_{m+1-k} \left[\frac{B_{n+k+1}(N) - B_{n+k+1}}{n+k+1}\right] \\
&= \sum_{k=0}^{m} \sum_{l=0}^{p-1-k} \sum_{q=0}^{p-k-l} \frac{B_k B_l}{(m+1)(p-k)} \binom{m+1}{k} \binom{p-k}{l} \binom{p-k-l}{q} (N-1)^q
\end{aligned}$$

Remarks

- The computation rules for $\mathcal{H}_{1,\dots,k}$ is reminiscent of the chain rule for differentiation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f_r \circ \cdots \circ f_1(x)\right) = f_r'\left(f_{r-1} \circ \cdots \circ f_1\right) \cdots f_1'(x).$$

- for r polynomials $P_1, \dots, P_r$ without constant terms,

$$\sum_{N > i_1 > \cdots > i_r > 0} P_1(i_1) \dots P_r(i_r) = \prod_{k=1}^{r} P_k\left(\mathcal{H}_{1,\dots,k}\right)$$

- Recurrence

$$H_{-n_1,\dots,-n_r}(N) = \frac{1}{n_r+1} \sum_{k=1}^{n_r+1} \binom{n_r+1}{k} B_{n_r+1-k} H_{-n_1,\dots,-n_{r-1}-k}(N)$$

Remarks

- Generating function

$$\mathcal{F}_r\left(w_1,\ldots,w_r;N\right)=\sum_{n_1,\ldots,n_r=0}^{\infty}\frac{w_1^{n_1}\ldots w_r^{n_r}}{n_1!\ldots n_r!}H_{-n_1,\ldots,-n_r}\left(N\right)$$

$$\mathcal{F}_r\left(w_1,\ldots,w_r;N\right)=\frac{\mathcal{F}_{r-1}\left(w_1,\ldots,w_{r-1}+w_r;N\right)-\mathcal{F}_{r-1}\left(w_1,\ldots,w_{r-1};N\right)}{e^{w_r}-1},$$

where

$$\mathcal{F}_1\left(w_1;N\right)=\frac{e^{Nw_1}-1}{e^{w_1}-1}.$$

- S-Sum:

$$S_{-n_1,\ldots,-n_r}\left(N\right)=\sum_{N\geq i_1\geq\cdots\geq i_r\geq 1}i_1^{n_1}\cdots i_r^{n_r}$$

$$S_{-n_1,\ldots,-n_r}\left(N\right)=\prod_{k=1}^{r}\bar{\mathcal{H}}_{1,\ldots,k}^{n_k},$$

where $\bar{\mathcal{H}}_1=\mathcal{H}\left(N+1\right)$ and $\bar{\mathcal{H}}_{1,\ldots k}=\mathcal{H}\left(\bar{\mathcal{H}}_{1,\ldots,k-1}+1\right)$.

Remarks

- Extended Bernoulli polynomials

$$B_{n+1}(x+1) - B_{n+1}(x) = (n+1)x^n$$

$$B_{n_1,\ldots,n_r}(z+1) = B_{n_1,\ldots,n_r}(z) + n_1 z^{n_1-1} B_{n_2,\ldots,n_r}(z)$$

Theorem (G. H. E. Duchamp, V. H. N. Minh and N. Q. Hoan)

$$H_{-n_1,\ldots,-n_r}(N) = \frac{B_{n_1+1,\ldots,n_r+1}(N+1) - \sum_{k=1}^{r-1} b'_{n_k+1,\ldots,n_r+1} B_{n_1+1,\ldots,n_k+1}(N+1)}{\prod_{i=1}^{r}(n_i+1)}.$$

- Theorem (L. J and C. Vignat)

$$\beta_{n_1,\ldots,n_r}(z) := B_{n_1,\ldots,n_r}(z) - B_{n_1,\ldots,n_r}(0)$$
$$= \left(\prod_{k=1}^{r} \frac{\partial}{\partial \mathcal{B}_k}\right) H_{-n_1,\ldots,-n_r}(z)$$

$$(\mathcal{H}(z))^n = \frac{B_{n+1}(z) - B_{n+1}}{n+1} = \int_0^z (\mathcal{B}+x)^n \, \mathrm{d}x$$

Generating Function

Let

$$\beta_{n_1,\ldots,n_r}(z) = B_{n_1,\ldots,n_r}(z) - B_{n_1,\ldots,n_r}(0),$$

then,

$$\beta_{n_1,\ldots,n_r}(z+1) = \beta_{n_1,\ldots,n_r}(z) + n_1 z^{n_1-1}\beta_{n_2,\ldots,n_r}(z).$$

Consider the generating function

$$\mathcal{G}_r(w_1,\ldots,w_r;z) = \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{w_1^{n_1}\ldots w_r^{n_2}}{n_1!\ldots n_r!}\beta_{n_1,\ldots,n_r}(z)$$

Then,

$$\mathcal{G}_r(w_{\mathbf{1}},\ldots,w_r;z) = \frac{w_r\left(\frac{w_{r-\mathbf{1}}}{w_{r-\mathbf{1}}+w_r}\mathcal{G}_{r-\mathbf{1}}(w_{\mathbf{1}},\ldots,w_{r-\mathbf{2}},w_{r-\mathbf{1}}+w_r;z) - \mathcal{G}_{r-\mathbf{1}}(w_{\mathbf{1}},\ldots,w_{r-\mathbf{2}},w_{r-\mathbf{1}};z)\right)}{e^{w_r}-\mathbf{1}},$$

where,

$$\mathcal{G}_1(w_1;z) = \frac{w\left(e^{zw}-1\right)}{e^{w}-1}.$$

Current Work

$$
\begin{aligned}
\mathrm{Li}_{-n_1,\ldots,-n_r}(z) &= \sum_{i_1>\cdots>i_r>0} i_1^{n_1}\cdots i_r^{n_r} z^{i_1} \\
&= \sum_{i_1>0} i_1^{n_1} z^{i_1} \left(\sum_{i_1>i_2>\cdots>i_r>0} i_2^{n_2}\cdots i_r^{n_r} \right) \\
&= \sum_{i_1>0} i_1^{n_1} z^{i_1} H_{-n_2,\ldots,-n_r}(i_1) \\
&= \sum_{i_1>0} i_1^{n_1} z^{i_1} \left[\mathcal{H}(i_1)\right]^{n_2} \left[\mathcal{H}\left(\mathcal{H}(i_1)\right)\right]^{n_3} \cdots \left[\mathcal{H}\left(\cdots\mathcal{H}(i_1)\right)\right]^{n_r}
\end{aligned}
$$

Current Work

$$\zeta_r(-n_1,\ldots,-n_r)=\prod_{k=1}^{r}(-1)^{n_k}\,\mathcal{C}_{1,\ldots,k}^{n_k+1}????H_{-n_1,\ldots,-n_r}(N)=\prod_{k=1}^{r}\mathcal{H}_{1,\ldots,k}^{n_k}$$

End

Thank you