Matrix Representation for Multiplicative Nested Sums

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Dalhousie Number Theory Seminar

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Acknowledgment

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- Diane Shi

Objects

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$$\mathcal{S}\left(f_1,\ldots,f_k;N,m\right):=\sum_{N\geq n_1\geq\cdots\geq n_k\geq m} f_1\left(n_1\right)\cdots f_k\left(n_k\right)$$

and

$$\mathcal{A}\left(f_1,\ldots,f_k;N,m\right):=\sum_{N>n_1>\cdots>n_k\geq m} f_1\left(n_1\right)\cdots f_k\left(n_k\right)$$

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$$\left.\begin{array}{c} k=1 \\ m=1 \\ N=\infty \\ f_1(n)=\frac{1}{n^s} \end{array}\right\} \Rightarrow \mathcal{S}\left(f_1;\infty,1\right)=\mathcal{A}\left(f_1;\infty,1\right)=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

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Let $m=1$, $N=\infty$, and $f_l(x)=1/x^{s_l}$ for $l=1,\ldots,k$. Then,

$$\mathcal{S}\left(f_1,\ldots,f_k;\infty,1\right):=\sum_{n_1\geq\cdots\geq n_k\geq 1}\frac{1}{n_1^{s_1}\cdots n_k^{s_k}}=\zeta^*\left(s_1,\ldots,s_k\right),$$

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Let $m = 1$ and

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Theorem. [K. Dilcher]

$$\sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1 \cdots n_k} = \sum_{\ell=1}^{N} (-1)^{\ell-1} \binom{N}{\ell} \frac{1}{\ell^k}.$$

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Example. $k = 2$, $N = 3$:

$$\sum_{3 \geq n_1 \geq n_2 \geq 1} = \frac{1}{1 \cdot 1} + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 1} + \frac{1}{2 \cdot 2} + \frac{1}{3 \cdot 2} + \frac{1}{3 \cdot 3}$$

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Outline

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Theorem. [L. Jiu and D. Shi]

$$\sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1 \cdots n_k} = N \cdot \left(\mathbf{S}_N^{k+1}\right)_{N,1}.$$

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$$\sum_{2 \geq n_1 \geq n_2 \geq 1} \frac{1}{n_1 n_2} = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{8}.$$

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For example, suppose we are at site "6":

$$\begin{array}{cccccccccc} & & & & & \text{here} & & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & & N \end{array}$$

Then, the next step only allows to walk to sites $\{1, 2, 3, 4, 5, 6\}$,

$$\mathbb{P}\,(6 \to 6) = \mathbb{P}\,(6 \to 5) = \mathbb{P}\,(6 \to 4) = \mathbb{P}\,(6 \to 3) = \mathbb{P}\,(6 \to 2) = \mathbb{P}\,(6 \to 1) = \frac{1}{6}.$$

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Focus on $\mathbb{P}\,(n_{k+1} = 1)$:

$$\mathbb{P}\,(n_{k+1} = 1) = \sum_{\text{All possible paths}} \frac{1}{N} \frac{1}{n_1} \cdots \frac{1}{n_k} = \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{N n_1 \cdots n_k}.$$

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On the other hand, the transition matrix of sites $\{1, \ldots, N\}$ is exactly given by $\mathbf{S}_N$, i.e,

$$\mathbf{S}_N = (\alpha_{i,j}) \text{ with } \alpha_{i,j} = \mathbb{P}(i \to j) = \begin{cases} 1/i, & \text{if } j \leq i; \\ 0, & \text{if } j > i. \end{cases}$$

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Therefore, after $k+1$ steps, entries of $\mathbf{S}_N^{k+1}$ give the transition probabilities among sites. In particular,

$$\left(\mathbf{S}_N^{k+1}\right)_{N,1} = \mathbb{P}(n_{k+1} = 1) = \frac{1}{N} \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1 \cdots n_k}.$$

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Remark.

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Main Results

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Theorem. [L. Jiu and D. Shi] Define, for $l = 1, \ldots, k$,

$$\mathbf{P}_N = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix},$$

$$\mathbf{S}_{N|f_l} := \begin{pmatrix} f_l(1) & 0 & 0 & \cdots & 0 \\ f_l(2) & f_l(2) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_l(N) & f_l(N) & f_l(N) & \cdots & f_l(N) \end{pmatrix},$$

and

$$\mathbf{A}_{N|f_l} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ f_l(1) & 0 & 0 & \cdots & 0 & 0 \\ f_l(2) & f_l(2) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_l(N-1) & f_l(N-1) & f_l(N-1) & \cdots & f_l(N-1) & 0 \end{pmatrix}.$$

Main Results (Continued)

Then, it holds that

$$\mathcal{S}\left(f_1, \ldots, f_k; N, m\right) = \left(\mathbf{P}_N \cdot \prod_{l=1}^{k} \mathbf{S}_{N|f_l}\right)_{N,m},$$

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Proposition. [L. Jiu and D. Shi]

$$\mathbf{A}_{(N-1)|f} = \left(\delta_{i-1,j}\right)_{N\times N} \mathbf{S}_{N|f}.$$

Main Results (Continued)

Then, it holds that

$$\mathcal{S}\left(f_1, \ldots, f_k; N, m\right) = \left(\mathbf{P}_N \cdot \prod_{l=1}^{k} \mathbf{S}_{N|f_l}\right)_{N,m},$$

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If $f(1), \ldots, f(N)$ are distinct,

$$\mathbf{S}_{N|f} = \mathbf{D}_{N|f} \operatorname{diag}\left(f(1), \ldots, f(N)\right) \left(\mathbf{D}_{N|f}\right)^{-1},$$

where

$$\mathbf{D}_{N|f} = \left(a_{i,j}\right)_{N\times N} \quad \text{and} \quad \left(\mathbf{D}_{N|f}\right)^{-1} = \left(b_{i,j}\right)_{N\times N}$$

Main Results (Continued)

$$a_{i,j} := \frac{f(i)}{f(N)} \prod_{k=i+1}^{N} \left(1 - \frac{f(k)}{f(j)}\right),$$

and

$$b_{i,j} = \begin{cases} 0, & \text{if } i < j; \\ \frac{f(N)}{f(i)} \prod_{\substack{k=j \\ k \neq i}}^{N} \frac{1}{1 - \frac{f(k)}{f(i)}}, & \text{if } i \geq j. \end{cases}$$

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Remark.

$$a_{N,j} = 1$$

Example

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Recall

$$f(x) = \frac{1}{x} \Rightarrow \mathbf{S}_{N|f} := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \end{pmatrix}$$

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$$\begin{aligned} & \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1 \cdots n_k} \\ = & N \cdot \left(\mathbf{S}_{N|f}^{k+1} \right)_{N,1} \\ = & N \cdot \left[\mathbf{D}_{N|f} \operatorname{diag} \left(f(1)^{k+1}, \ldots, f(N)^{k+1} \right) \left(\mathbf{D}_{N|f} \right)^{-1} \right]_{N,1} \end{aligned}$$

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&= \sum_{j=1}^{N} (-1)^{\ell-1} \binom{N}{\ell} \frac{1}{\ell^k}.
\end{aligned}$$

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Butler and Karasik obtained if $G(n,k)$ satisfies $G(n,n) = 1$, $G(n,-k) = 0$ and for $k \geq 1$

$$G(n,k) = G(n-1,k-1) + g(k)G(n-1,k),$$

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A note on nested sums, S. Butler and P. Karasik, *J. Integer Seq.* **13** (2010), Article 10.4.4.

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- When $k = 1$, an induction on N shows directly that

$$\begin{aligned}\sum_{N \geq n_1 \geq 1} g(n_1) &= g(N) + G(N, N-1) \\ &= g(N) G(N, N) + G(N, N-1) \\ &= G(N+1, N).\end{aligned}$$

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- For inductive step in k, similarly from, we see

$$S\left(\underbrace{g, \ldots, g}_{k}, N, 1\right) = g(N)\left(\mathbf{S}^{k}_{N|g}\right)_{N,1} = g(N)\left(\mathbf{S}_{N|g}\left(\mathbf{S}^{k-1}_{N|g}\right)\right)_{N,1}$$
$$= \frac{1}{g(N)} \sum_{m=1}^{N} g(N) \cdot g(m) \cdot G(m+k-1, m)$$
$$= G(N+k, N),$$

by recurrence.

$N = \infty$

N=∞

Let $m = 1$, $N = \infty$, and $f_l(x) = 1/x^{s_l}$ for $l = 1, \ldots, k$. Then,

$$\mathcal{S}\left(f_1, \ldots, f_k; \infty, 1\right) := \sum_{n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}} = \zeta^*\left(s_1, \ldots, s_k\right),$$

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Example.

$$\zeta^*\left(2, 1\right) = 2\zeta(3).$$

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$$\zeta^*(s_1, s_2) = \zeta(s_1, s_2) + \zeta(s_1 + s_2),$$
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Theorem. [L. Jiu and D. Shi]

$$\mathcal{S}\,(f, g; N-1, m) = \mathcal{A}\,(f, g; N, m) + \mathcal{A}\,(fg; N, m)$$

and

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KEY:

$$\mathbf{A}_{(N-1)|f} = \left(\delta_{i-1,j}\right)_{N\times N} \mathbf{S}_{N|f}.$$

Open Question

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Theorem. [M. Hoffman] For any real $i_1, \ldots, i_k > 1$,

$$\sum_{\sigma \in S_k} \zeta^* \left(i_{\sigma(1)}, \ldots, i_{\sigma(k)} \right) = \sum_{\text{partitions } \Pi \text{ of } \{1,\ldots,k\}} c\left(\Pi\right) \zeta\left(i_1, \ldots, i_k, \Pi\right),$$

where $\Pi = \{P_1, \ldots, P_\ell\}$ is a set partition of $\{1, \ldots, k\}$,

$$c\left(\Pi\right) := (|P_1| - 1)! \cdots (|P_\ell| - 1)!,$$

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And

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where

$$\tilde{c}(\Pi) = (-1)^{k-\ell} c(\Pi).$$

$k \to \infty$

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$$\lim_{k\to\infty} \sum_{N\geq n_1\geq\cdots\geq n_k\geq 1} \frac{1}{n_1\cdots n_k} = N.$$

Proof.

$$\sum_{N\geq n_1\geq\cdots\geq n_k\geq 1} \frac{1}{n_1\cdots n_k} = N\cdot\mathbb{P}\left(n_{k+1}=1\right).$$

$\square$