~~Two~~ Three Examples on Computer Proofs of Combinatorial ~~Identities~~ Results

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Outline

Example 1

Example 2

Example 3

Example1

Question

Example1

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$$1^2 + 2^2 + \cdots + n^2 = \sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1). \qquad (*)$$

Example1

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$$1^2 + 2^2 + \cdots + n^2 = \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1). \qquad (*)$$

1. *Induction;*

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$$1^2 + 2^2 + \cdots + n^2 = \sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1). \qquad (*)$$

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- $n = 1$:

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$$1^2 + 2^2 + \cdots + n^2 = \sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1). \qquad (*)$$

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- $n = 1$:

$$LHS = 1 = \frac{1 \cdot (1+1) \cdot (2+1)}{6} = RHS;$$

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Example1

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$$1^2 + 2^2 + \cdots + n^2 = \sum_{k=1}^{n} k^2 = ? \qquad (*)$$

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Example1

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2. *Observation*:
   - $n^3 - (n-1)^3 = 3n^2 - 3n + 1$;

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   - $2^3 - 1^3 = 3 \cdot 2^2 - 3 \cdot 2 + 1 (= 7)$ and
   - $1^3 - 0^3 = 3 \cdot 1^2 - 3 \cdot 1 + 1.$

$$n^3 = 3\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k + n = 3\sum_{k=1}^{n} k^2 + \frac{n(n+1)}{2} + n.$$

$$\Rightarrow \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

create telescoping

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$$1^2 + 2^2 + \cdots + n^2 = \sum_{k=1}^{n} k^2 = ? \qquad (*)$$

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$$n^3 = 3 \sum_{k=1}^{n} k^2 - 3 \sum_{k=1}^{n} k + n$$

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$$n^3 = 3 \sum_{k=1}^{n} k^2 - 3 \sum_{k=1}^{n} k + n = 3 \sum_{k=1}^{n} k^2 - \frac{3n(n+1)}{2} + n.$$

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$$1^2 + 2^2 + \cdots + n^2 = \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1). \qquad (*)$$

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- $n = 1$: $LHS = 1 = RHS$;

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- $n = 1$: $LHS = 1 = RHS$;
- $n = 2$: $LHS = 5 = RHS$;

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- $n = 1$: $LHS = 1 = RHS$;
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- $n = 3$: $LHS = 14 = RHS$;

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   - $n = 1$: $LHS = 1 = RHS$;
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Theorem. For any positive integer n,

$$f(n) = 1^2 + 2^2 + \cdots + n^2$$

is a polynomial in variable n, of degree 3.

Example1

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$$f(n) = 1^2 + 2^2 + \cdots + n^2 = \alpha n^3 + \beta n^2 + \gamma n + \delta$$

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Example1

Example1

$$\begin{cases} \alpha + \beta + \gamma + \delta & = 1 \\ 8\alpha + 4\beta + 2\gamma + \delta & = 5 \\ 27\alpha + 9\beta + 3\gamma + \delta & = 14 \\ 64\alpha + 16\beta + 4\gamma + \delta & = 30 \end{cases} \Rightarrow \begin{cases} \alpha = 1/3 \\ \beta = 1/6 \\ \gamma = 1/2 \\ \delta = 0 \end{cases}$$

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Proof of the Theorem.

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Theorem. For any positive integers d and n,

$$Q(n) := 1^d + 2^d + \cdots + n^d = \sum_{k=1}^{n} k^d$$

is a polynomial in variable n of degree $d+1$.

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$$Q(n) = \alpha_{d+1} n^{d+1} + \alpha_d n^d + \cdots + \alpha_1 d + \alpha_0.$$

Example1

Theorem. Let $P_d(x)$ be a polynomial of degree d. Define

$$Q(n) := P_d(1) + P_d(2) + \cdots + P_d(n) = \sum_{k=1}^{n} P_d(k).$$

Then, $Q(n)$ is a polynomial in variable n of degree $d + 1$.

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$$1 + 2 + \cdots + n = \frac{n\,(n+1)}{2}$$

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$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Example2

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$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}. \qquad (**)$$

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$$LHS = \sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k}.$$

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Generating function proof.

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$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j.$$

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$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j.$$

$$\sum_{j=0}^{2n} \binom{2n}{j} x^j = \underline{(1+x)^{2n} = (1+x)^n \cdot (1+x)^n} = \sum_{j=0}^{2n} \sum_{k=0}^{j} \binom{n}{k} \binom{n}{j-k} x^j$$

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Consider the term of $j = n$ (the coefficients of x^n on both sides)

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Find $G(n,k)$ *such that*

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$$

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Find $G(n,k)$ such that

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Recall in Question 1

$$k^2 = \left[\frac{(k+1)^3 - \frac{3}{2}(k+1)^2 + \frac{k+1}{2}}{3}\right] - \left[\frac{k^3 - \frac{3}{2}k^2 + \frac{k}{2}}{3}\right]$$

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- Since $\binom{n}{k} = 0$ when $k < 0$ or $k > n$,

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$$\sum_{k \in \mathbb{Z}} \binom{n}{k}^2$$

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- Since $\binom{n}{k} = 0$ when $k < 0$ or $k > n$,

$$\sum_{k \in \mathbb{Z}} \binom{n}{k}^2 \Rightarrow F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$$

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We can simply sum for $k \in \mathbb{Z}$ so that the left hand side becomes

$$f(n+1) - f(n).$$

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Other wise, we need to sum for k from 0 to $n+1$, giving

$$f(n+1) - [f(n) + F(n,n+1)].$$

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$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$$

- What about

$$\lim_{k\to-\infty} G(n,k) \quad \text{and} \quad \lim_{k\to+\infty} G(n,k)?$$

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$$G(n,k) = F(n,k)R(n,k)$$

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- What about

$$\lim_{k\to-\infty} G(n,k) \quad \text{and} \quad \lim_{k\to+\infty} G(n,k)?$$

$$G(n,k) = F(n,k)R(n,k) = F(n,k)\cdot\frac{P(n,k)}{Q(n,k)} \quad \text{for polynomials } P \,\&\, Q.$$

Example2

Question

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}. \quad (**)$$

Example2

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$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}. \qquad (**)$$

Step 1.

$$\sum_{k=0}^{n} \frac{\binom{n}{k}^2}{\binom{2n}{n}} = 1$$

Example2

Question

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}. \quad (**)$$

<u>*Step* 1</u>.

$$\sum_{k=0}^{n} \frac{\binom{n}{k}^2}{\binom{2n}{n}} = 1 = \sum_{k\in\mathbb{Z}} \frac{(n!)^4}{(k!)^2\left((n-k)!\right)^2 (2n)!}.$$

Example2

Question

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}. \quad (**)$$

<u>*Step 1*</u>.

$$\sum_{k=0}^{n} \frac{\binom{n}{k}^2}{\binom{2n}{n}} = 1 = \sum_{k\in\mathbb{Z}} \frac{(n!)^4}{(k!)^2\,((n-k)!)^2\,(2n)!}.$$

$$F(n,k) = \frac{(n!)^4}{(k!)^2\,((n-k)!)^2\,(2n)!}.$$

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Question

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}. \quad (**)$$

<u>*Step 1*</u>*.*

$$\sum_{k=0}^{n} \frac{\binom{n}{k}^2}{\binom{2n}{n}} = 1 = \sum_{k \in \mathbb{Z}} \frac{(n!)^4}{(k!)^2\left((n-k)!\right)^2 (2n)!}.$$

$$F(n,k) = \frac{(n!)^4}{(k!)^2\left((n-k)!\right)^2 (2n)!}.$$

<u>*Step 2*</u>*. Find $R(n,k)$*

Example2

Question

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}. \qquad (**)$$

Step 1.

$$\sum_{k=0}^{n} \frac{\binom{n}{k}^2}{\binom{2n}{n}} = 1 = \sum_{k\in\mathbb{Z}} \frac{(n!)^4}{(k!)^2\left((n-k)!\right)^2(2n)!}.$$

$$F(n,k) = \frac{(n!)^4}{(k!)^2\left((n-k)!\right)^2(2n)!}.$$

Step 2. Find $R(n,k)$

$$R(n,k) = \frac{(2k-3n-3)\,k^2}{2(n+1-k)^2(2n+1)}$$

Example2

$$F(n,k) = \frac{(n!)^4}{(k!)^2\,((n-k)!)^2\,(2n)!} \quad \text{and} \quad R(n,k) = \frac{(2k-3n-3)\,k^2}{2(n+1-k)^2(2n+1)}$$

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- $$F(n+1,k) - F(n,k)$$

Example2

$$F(n,k) = \frac{(n!)^4}{(k!)^2\,((n-k)!)^2\,(2n)!} \quad \text{and} \quad R(n,k) = \frac{(2k-3n-3)\,k^2}{2(n+1-k)^2(2n+1)}$$

- $$\begin{aligned} & F(n+1,k) - F(n,k) \\ = \; & \frac{((n+1)!)^4}{(k!)^2\,((n+1-k)!)^2\,(2n+2)!} - \frac{(n!)^4}{(k!)^2\,((n-k)!)^2\,(2n)!} \end{aligned}$$

Example2

$$F(n,k) = \frac{(n!)^4}{(k!)^2\,((n-k)!)^2\,(2n)!} \quad \text{and} \quad R(n,k) = \frac{(2k-3n-3)\,k^2}{2(n+1-k)^2(2n+1)}$$

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Example2

$$F(n,k) = \frac{(n!)^4}{(k!)^2\,((n-k)!)^2\,(2n)!} \quad \text{and} \quad R(n,k) = \frac{(2k-3n-3)\,k^2}{2(n+1-k)^2(2n+1)}$$

-

$$\begin{aligned} & F(n+1,k) - F(n,k) \\ = \quad & \frac{((n+1)!)^4}{(k!)^2\,((n+1-k)!)^2\,(2n+2)!} - \frac{(n!)^4}{(k!)^2\,((n-k)!)^2\,(2n)!} \\ = \quad & \frac{(n!)^4}{(k!)^2\,((n-k)!)^2\,(2n)!}\left[\frac{(n+1)^4}{(n+1-k)^2\,(2n+2)\,(2n+1)} - 1\right] \end{aligned}$$

Example2

$$F(n,k) = \frac{(n!)^4}{(k!)^2\,((n-k)!)^2\,(2n)!} \quad \text{and} \quad R(n,k) = \frac{(2k-3n-3)\,k^2}{2(n+1-k)^2(2n+1)}$$

- $$\begin{aligned} & F(n+1,k) - F(n,k) \\ = \; & \frac{((n+1)!)^4}{(k!)^2\,((n+1-k)!)^2\,(2n+2)!} - \frac{(n!)^4}{(k!)^2\,((n-k)!)^2\,(2n)!} \\ = \; & \frac{(n!)^4}{(k!)^2\,((n-k)!)^2\,(2n)!}\left[\frac{(n+1)^4}{(n+1-k)^2\,(2n+2)\,(2n+1)} - 1\right] \\ = \; & F(n,k)\frac{3n^3 + (7-8k)\,n^2 + \left(2k^2+5\right)n + 2k^2 - 4k + 1}{2\,(n+1-k)^2\,(2n+2)} \end{aligned}$$

Example2

$$F(n,k) = \frac{(n!)^4}{(k!)^2\,((n-k)!)^2\,(2n)!} \quad \text{and} \quad R(n,k) = \frac{(2k-3n-3)\,k^2}{2(n+1-k)^2(2n+1)}$$

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$$G(n,k+1) - G(n,k)$$

Example2

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$$\begin{aligned} & G(n,k+1) - G(n,k) \\ = \quad & F(n,k+1)R(n,k+1) - F(n,k)R(n,k) \end{aligned}$$

Example2

$$F(n,k) = \frac{(n!)^4}{(k!)^2\,((n-k)!)^2\,(2n)!} \quad \text{and} \quad R(n,k) = \frac{(2k-3n-3)\,k^2}{2(n+1-k)^2(2n+1)}$$

$$\begin{aligned} & G(n,k+1) - G(n,k) \\ = \quad & F(n,k+1)R(n,k+1) - F(n,k)R(n,k) \\ = \quad & F(n,k)\left[\frac{k^2(3n+3-2k)}{2(n+1-k)^2(2n+1)} - \frac{(3n+1-2k)}{2(2n+1)}\right] \end{aligned}$$

Example2

$$F(n,k)=\frac{(n!)^4}{(k!)^2\left((n-k)!\right)^2(2n)!} \quad \text{and} \quad R(n,k)=\frac{(2k-3n-3)\,k^2}{2(n+1-k)^2(2n+1)}$$

$$\begin{aligned} & G(n,k+1)-G(n,k) \\ = \quad & F(n,k+1)R(n,k+1)-F(n,k)R(n,k) \\ = \quad & F(n,k)\left[\frac{k^2(3n+3-2k)}{2(n+1-k)^2(2n+1)}-\frac{(3n+1-2k)}{2(2n+1)}\right] \\ = \quad & F(n,k)\frac{3n^3+(7-8k)\,n^2+\left(2k^2+5\right)n+2k^2-4k+1}{2\,(n+1-k)^2\,(2n+2)} \end{aligned}$$

Example2

$$F(n,k)=\frac{(n!)^4}{(k!)^2\left((n-k)!\right)^2(2n)!} \quad \text{and} \quad R(n,k)=\frac{(2k-3n-3)\,k^2}{2(n+1-k)^2(2n+1)}$$

$$\begin{aligned}
& G(n,k+1)-G(n,k) \\
= \quad & F(n,k+1)R(n,k+1)-F(n,k)R(n,k) \\
= \quad & F(n,k)\left[\frac{k^2(3n+3-2k)}{2(n+1-k)^2(2n+1)}-\frac{(3n+1-2k)}{2(2n+1)}\right] \\
= \quad & F(n,k)\frac{3n^3+(7-8k)\,n^2+\left(2k^2+5\right)n+2k^2-4k+1}{2\,(n+1-k)^2\,(2n+2)}
\end{aligned}$$

$$F(n+1,k)-F(n,k)=G(n,k+1)-G(n,k)$$

Example2

$$G(n,k) = F(n,k)R(n,k) = \frac{(2k-3n-3)\,k^2}{2(n+1-k)^2(2n+1)} \cdot \frac{\binom{n}{k}^2}{\binom{2n}{n}}$$

Example2

$$G(n,k) = F(n,k)R(n,k) = \frac{(2k-3n-3)\,k^2}{2(n+1-k)^2(2n+1)} \cdot \frac{\binom{n}{k}^2}{\binom{2n}{n}}$$

$$\lim_{k\to-\infty} G(n,k) = 0 = \lim_{k\to+\infty} G(n,k)$$

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$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$$

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$$G(n,k) = F(n,k)R(n,k) = \frac{(2k-3n-3)\,k^2}{2(n+1-k)^2(2n+1)} \cdot \frac{\binom{n}{k}^2}{\binom{2n}{n}}$$

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$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$$

$$f(n+1) - f(n) = 0$$

Example2

$$G(n,k) = F(n,k)R(n,k) = \frac{(2k-3n-3)\,k^2}{2(n+1-k)^2(2n+1)} \cdot \frac{\binom{n}{k}^2}{\binom{2n}{n}}$$

$$\lim_{k \to -\infty} G(n,k) = 0 = \lim_{k \to +\infty} G(n,k)$$

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$$

$$f(n+1) - f(n) = 0$$

$$f(n) = \sum_{k=0}^{n} \frac{\binom{n}{k}^2}{\binom{2n}{n}}$$

Example2

$$G(n,k) = F(n,k)R(n,k) = \frac{(2k-3n-3)\,k^2}{2(n+1-k)^2(2n+1)} \cdot \frac{\binom{n}{k}^2}{\binom{2n}{n}}$$

$$\lim_{k\to-\infty} G(n,k) = 0 = \lim_{k\to+\infty} G(n,k)$$

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$$

$$f(n+1) - f(n) = 0$$

$$f(n) = \sum_{k=0}^{n} \frac{\binom{n}{k}^2}{\binom{2n}{n}} = f(0)$$

Example2

$$G(n,k) = F(n,k)R(n,k) = \frac{(2k-3n-3)\,k^2}{2(n+1-k)^2(2n+1)} \cdot \frac{\binom{n}{k}^2}{\binom{2n}{n}}$$

$$\lim_{k\to-\infty} G(n,k) = 0 = \lim_{k\to+\infty} G(n,k)$$

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$$

$$f(n+1) - f(n) = 0$$

$$f(n) = \sum_{k=0}^{n} \frac{\binom{n}{k}^2}{\binom{2n}{n}} = f(0) = \frac{\binom{0}{0}^2}{\binom{0}{0}} = 1$$

Example2

$$G(n,k) = F(n,k)R(n,k) = \frac{(2k-3n-3)\,k^2}{2(n+1-k)^2(2n+1)} \cdot \frac{\binom{n}{k}^2}{\binom{2n}{n}}$$

$$\lim_{k\to-\infty} G(n,k) = 0 = \lim_{k\to+\infty} G(n,k)$$

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$$

$$f(n+1) - f(n) = 0$$

$$f(n) = \sum_{k=0}^{n} \frac{\binom{n}{k}^2}{\binom{2n}{n}} = f(0) = \frac{\binom{0}{0}^2}{\binom{0}{0}} = 1$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Example2

$$F(n,k) = \frac{\binom{n}{k}^2}{\binom{2n}{n}} \quad \text{and} \quad R(n,k) = \frac{(2k-3n-3)\,k^2}{2(n+1-k)^2(2n+1)}$$

Example2

$$F(n,k)=\frac{\binom{n}{k}^2}{\binom{2n}{n}} \quad \text{and} \quad R(n,k)=\frac{(2k-3n-3)\,k^2}{2(n+1-k)^2(2n+1)}$$

$R(n,k)$ is called WZ proof certificate

Example2

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Example2

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$R(n,k)$ is called WZ proof certificate (Wilf–Zeilberger)

$$\left.\begin{aligned} \frac{F(n+1,k)}{F(n,k)} &= \frac{(n+1)^4}{(n+1-k)^2\,(2n+2)\,(2n+1)} \\ \frac{F(n,k+1)}{F(n,k)} &= \frac{(n-k)^2}{(k+1)^2} \end{aligned}\right\} \text{rational in } n \text{ \& } k.$$

Example2

$$F(n,k)=\frac{\binom{n}{k}^2}{\binom{2n}{n}} \quad \text{and} \quad R(n,k)=\frac{(2k-3n-3)\,k^2}{2(n+1-k)^2(2n+1)}$$

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$$\left.\begin{aligned}\frac{F(n+1,k)}{F(n,k)} &= \frac{(n+1)^4}{(n+1-k)^2\,(2n+2)\,(2n+1)}\\ \frac{F(n,k+1)}{F(n,k)} &= \frac{(n-k)^2}{(k+1)^2}\end{aligned}\right\}\text{rational in } n \,\&\, k.$$

6.3 How the algorithm works

The creative telescoping algorithm is for the fast discovery of the recurrence for a proper hypergeometric term, in the telescoped form (6.1.3). The algorithmic implementation makes strong use of the existence, but not of the method of proof used in the existence theorem.

More precisely, what we do is this. We now *know* that a recurrence (6.1.3) exists. On the left side of the recurrence there are unknown coefficients $a_0, \ldots, a_J$; on the right side there is an unknown function G; and the order J of the recurrence is unknown, except that bounds for it were established in the Fundamental Theorem (Theorem 4.4.1 on page 65).

We begin by fixing the assumed order J of the recurrence. We will then look for a recurrence of that order, and if none exists, we'll look for one of the next higher order.

For that fixed J, let's denote the left side of (6.1.3) by t_k, so that

$$t_k = a_0F(n,k) + a_1F(n+1,k) + \cdots + a_JF(n+J,k). \qquad (6.3.1)$$

Example2

Then we have for the term ratio

$$\frac{t_{k+1}}{t_k} = \frac{\sum_{j=0}^{J} a_j F(n+j,k+1)/F(n,k+1)}{\sum_{j=0}^{J} a_j F(n+j,k)/F(n,k)} \frac{F(n,k+1)}{F(n,k)}. \tag{6.3.2}$$

The second member on the right is a rational function of n, k, say

$$\frac{F(n,k+1)}{F(n,k)} = \frac{r_1(n,k)}{r_2(n,k)},$$

where the r's are polynomials, and also

$$\frac{F(n,k)}{F(n-1,k)} = \frac{s_1(n,k)}{s_2(n,k)},$$

say, where the s's are polynomials. Then

$$\frac{F(n+j,k)}{F(n,k)} = \prod_{i=0}^{j-1} \frac{F(n+j-i,k)}{F(n+j-i-1,k)} = \prod_{i=0}^{j-1} \frac{s_1(n+j-i,k)}{s_2(n+j-i,k)}. \tag{6.3.3}$$

It follows that

$$\begin{aligned} \frac{t_{k+1}}{t_k} &= \frac{\sum_{j=0}^{J} a_j \left\{\prod_{i=0}^{j-1} \frac{s_1(n+j-i,k+1)}{s_2(n+j-i,k+1)}\right\}}{\sum_{j=0}^{J} a_j \left\{\prod_{i=0}^{j-1} \frac{s_1(n+j-i,k)}{s_2(n+j-i,k)}\right\}} \frac{r_1(n,k)}{r_2(n,k)} \\ &= \frac{\sum_{j=0}^{J} a_j \left\{\prod_{i=0}^{j-1} s_1(n+j-i,k+1) \prod_{r=j+1}^{J} s_2(n+r,k+1)\right\}}{\sum_{j=0}^{J} a_j \left\{\prod_{i=0}^{j-1} s_1(n+j-i,k) \prod_{r=j+1}^{J} s_2(n+r,k)\right\}} \\ &\quad \times \frac{r_1(n,k)}{r_2(n,k)} \frac{\prod_{r=1}^{J} s_2(n+r,k)}{\prod_{r=1}^{J} s_2(n+r,k+1)}. \end{aligned} \tag{6.3.4}$$

Thus we have

$$\frac{t_{k+1}}{t_k} = \frac{p_0(k+1)}{p_0(k)} \frac{r(k)}{s(k)}, \tag{6.3.5}$$

where

$$p_0(k) = \sum_{j=0}^{J} a_j \left\{ \prod_{i=0}^{j-1} s_1(n+j-i,k) \prod_{r=j+1}^{J} s_2(n+r,k) \right\}, \tag{6.3.6}$$

and

$$r(k) = r_1(n,k) \prod_{r=1}^{J} s_2(n+r,k), \tag{6.3.7}$$

$$s(k) = r_2(n,k) \prod_{r=1}^{J} s_2(n+r,k+1). \tag{6.3.8}$$

Note that *the assumed coefficients* a_j *do not appear in* $r(k)$ *or in* $s(k)$, but only in $p_0(k)$.

Next, by Theorem 5.3.1, we can write $r(k)/s(k)$ in the canonical form

$$\frac{r(k)}{s(k)} = \frac{p_1(k+1)}{p_1(k)} \frac{p_2(k)}{p_3(k)}, \tag{6.3.9}$$

in which the numerator and denominator on the right are coprime, and

$$\gcd(p_2(k), p_3(k+j)) = 1 \quad (j = 0, 1, 2, \ldots).$$

Hence if we put $p(k) = p_0(k)p_1(k)$ then from eqs. (6.3.5) and (6.3.9), we obtain

$$\frac{t_{k+1}}{t_k} = \frac{p(k+1)}{p(k)} \frac{p_2(k)}{p_3(k)}. \tag{6.3.10}$$

This is now a standard setup for Gosper's algorithm (compare it with the discussion on page 76), and we see that t_k will be an indefinitely summable hypergeometric term if and only if the recurrence (compare eq. (5.2.6))

$$p_2(k)b(k+1) - p_3(k-1)b(k) = p(k) \tag{6.3.11}$$

has a polynomial solution $b(k)$.

The remarkable feature of this equation (6.3.11) is that *the coefficients* $p_2(k)$ *and* $p_3(k)$ *are independent of the unknowns* $\{a_j\}_{j=0}^{J}$, *and the right side* $p(k)$ *depends on them linearly.* Now watch what happens as a result. We look for a polynomial solution to (6.3.11) by first, as in Gosper's algorithm, finding an upper bound on the degree, say Δ, of such a solution. Next we assume $b(k)$ as a general polynomial of that degree, say

$$b(k) = \sum_{l=0}^{\Delta} \beta_l k^l,$$

with all of its coefficients to be determined. We substitute this expression for $b(k)$ in (6.3.11), and we find a system of simultaneous *linear* equations in the $\Delta + J + 2$ unknowns

$$a_0, a_1, \ldots, a_J, \beta_0, \ldots, \beta_\Delta.$$

The linearity of this system is directly traceable to the italicized remark above.

We then solve the system, if possible, for the a_j's and the β_l's. If no solution exists, then there is no recurrence of telescoped form (6.1.3) and of the assumed order J. In such a case we would next seek such a recurrence of order $J+1$. If on the other

Example2

hand a polynomial solution $b(k)$ of equation (6.3.11) does exist, then we will have found all of the a_j's of our assumed recurrence (6.1.3), and, by eq. (5.2.5) we will also have found the $G(n,k)$ on the right hand side, as

$$G(n,k) = \frac{p_3(k-1)}{p(k)} b(k) t_k. \qquad (6.3.12)$$

See Koornwinder [Koor93] for further discussion and a q-analogue.

Example2

6.4 Examples **109**

hand a polynomial solution $b(k)$ of equation (6.3.11) does exist, then we will have found all of the a_j's of our assumed recurrence (6.1.3), and, by eq. (5.2.5) we will also have found the $G(n,k)$ on the right hand side, as

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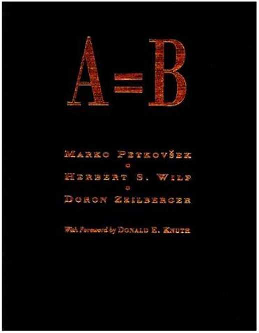
See Koornwinder [Koor93] for further discussion and a q-analogue.



Example2

6.4 Examples **109**

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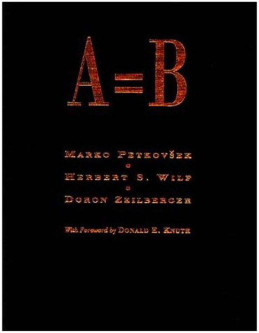
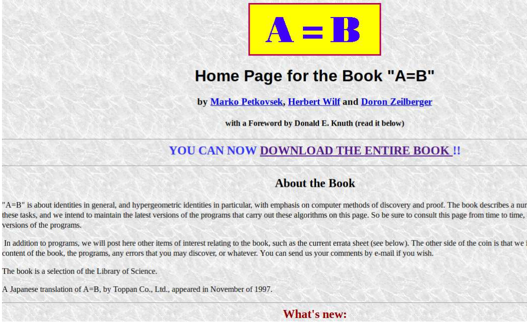


https://www.math.upenn.edu/~wilf/AeqB.html



Example3

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The unimodality of a polynomial coming from a rational integral. Back to the original proof

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ABSTRACT

A sequence of coefficients that appeared in the evaluation of a rational integral has been shown to be unimodal. An alternative proof is presented.

1. Introduction

The polynomial

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Example3

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The unimodality of a polynomial coming from a rational integral. Back to the original proof

Tewodros Amdeberhan, Atul Dixit, Xiao Guan, Lin Jiu, Victor H. Moll *

Department of Mathematics, Tulane University, New Orleans, LA 70118, United States

A R T I C L E I N F O

Article history:
Received 30 April 2013
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Keywords:
Hypergeometric function
Unimodal polynomials
Monotonicity

A B S T R A C T

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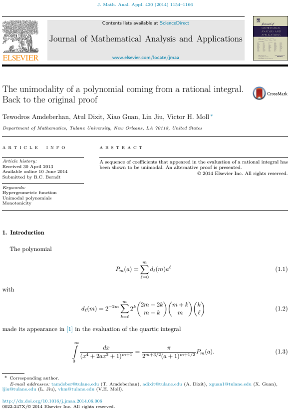

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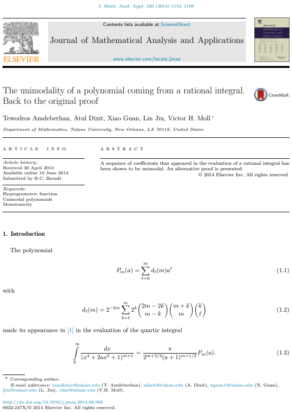

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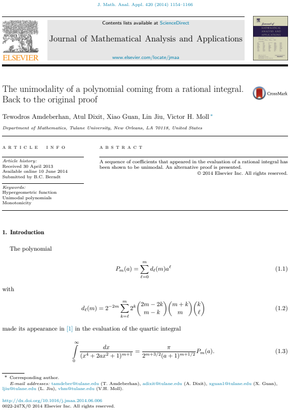
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Letting $p \to \infty$: LHS$\to 0$

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Letting $p \to \infty$: LHS$\to 0$ while $\lim_{n\to\infty} \frac{a_n}{c_n} = \frac{27}{16}$.

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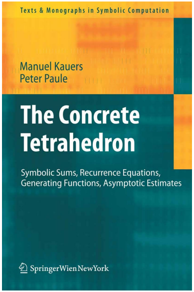

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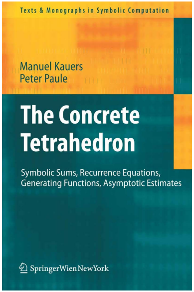
  - "The Concrete Tetrahedron" 

- There are several softwares: Sage, Maple, Mathematica, ...

Conclusion

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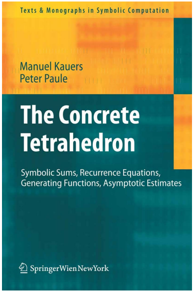
  - "The Concrete Tetrahedron" 

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Thank you!