Orthogonal Polynomials for Bernoulli and Euler Polynomials

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Dalhousie University
Number Theory Seminar

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Acknowledgment

- Diane Shi
- Tianjin University

Objects

- Bernoulli numbers B_n:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!};$$

- Bernoulli polynomial $B_n(x)$:

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!};$$

- Bernoulli polynomial of order p $B_n^{(p)}(x)$:

$$\left(\frac{t}{e^t - 1}\right)^p e^{xt} = \sum_{n=0}^{\infty} B_n^{(p)}(x) \frac{t^n}{n!};$$

- Euler numbers E_n:

$$\frac{2e^t}{e^{2t} + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!};$$

- Euler polynomial $E_n(x)$:

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$$B_n^{(1)}(x) = B_n(x);\ B_n(0) = B_n;\ E_n^{(1)}(x) = E_n(x);\ E_n(1/2) = E_n/2^n.$$

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2. Touchard [16, eq. 44] computed the monic orthogonal polynomials with respect to the B_n, denoted by $R_n(y)$:

$$R_{n+1}(y) = \left(y + \frac{1}{2}\right) R_n(y) + \frac{n^4}{4(2n+1)(2n-1)} R_{n-1}(y).$$

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- Generalization
- Probabilistic interpretations: Letting

$$p_B(t) := \frac{\pi}{2}\mathrm{sech}^2(\pi t) \quad \text{and} \quad p_E(t) := \mathrm{sech}(\pi t), \quad (t \in \mathbb{R})$$

  we define two random variables L_B and L_E with density functions p_B and p_E, respectively. Then, with $i^2 = -1$,

$$B_n(x) = \mathbb{E}\left[\left(iL_B + x - \frac{1}{2}\right)^n\right] = \int_{\mathbb{R}} \left(it + x - \frac{1}{2}\right)^n p_B(t)\mathrm{d}t, \quad [5, eq.\ 2.14]$$

$$E_n(x) = \mathbb{E}\left[\left(iL_E + x - \frac{1}{2}\right)^n\right] = \int_{\mathbb{R}} \left(it + x - \frac{1}{2}\right)^n p_E(t)\mathrm{d}t. \quad [9, eq.\ 2.3]$$

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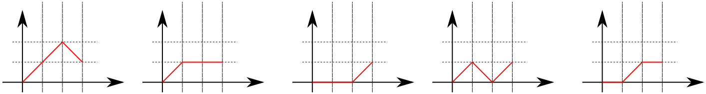

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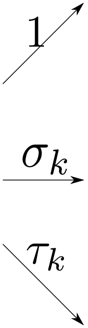

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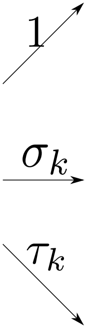

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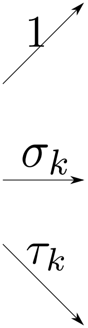

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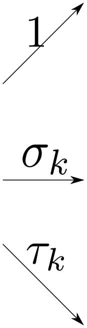

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Let X be an arbitrary random variable, with moments m_n and monic orthogonal polynomials $P_n(y)$ satisfying the recurrence

$$P_{n+1}(y) = (y - s_n)P_n(y) - t_n P_{n-1}(y).$$

Then, we have

$$\sum_{n=0}^{\infty} m_n z^n = \cfrac{m_0}{1 - s_0 z - \cfrac{t_1 z^2}{1 - s_1 z - \cfrac{t_2 z^2}{1 - s_2 z - \cdots}}}.$$

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By letting $(\sigma_k, \tau_k) = (s_k, t_k)$. If further assuming $m_0 = 1$, we have

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Example. The monic orthogonal polynomials with respect to E_n, $Q_n(y)$, satisfy

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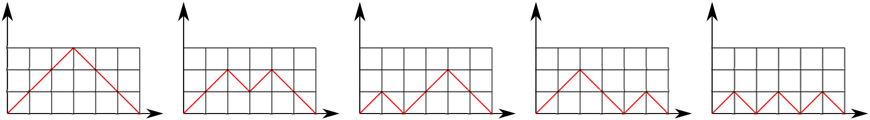

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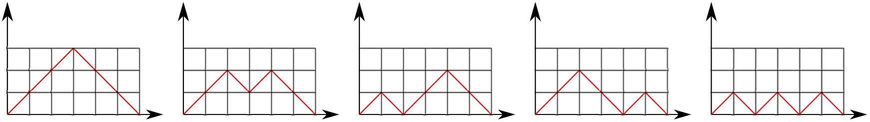

Then, by noting that each diagonally down path from (j, k) to $(j + 1, k - 1)$ has weight $-k^2$, we have

$$-61 = E_6 = (-1)^3 \left(3^2 2^2 1^2 + 2^2 2^2 1^2 + 1^2 2^2 1^2 + 2^2 1^2 1^2 + 1^2 1^2 1^2\right).$$

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Lemma. [L. Jiu and D. Shi]

random variable	moments	monic orthogoal polynomial
X	m_n	$P_n(y): \quad P_{n+1}(y) = (y - s_n)P_n(y) - t_n P_{n-1}(y)$
$X + c$	$\sum_{k=0}^{n} \binom{n}{k} m_k c^{n-k}$	$\bar{P}_n(y): \quad \bar{P}_{n+1}(y) = (y - s_n - c)\bar{P}_n(y) - t_n \bar{P}_{n-1}(y)$
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$$\bar{P}_n(y) := P_n(y - c) \quad \text{and} \quad \tilde{P}_n(y) := C^n P_n(y/C).$$

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E_n	$Q_{n+1}(y) = yQ_n(y) + n^2 Q_{n-1}(y)$
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$$\Delta_n(\mathbf{a}) := \det \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & a_3 & \cdots & a_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & a_{n+2} & \cdots & a_{2n} \end{pmatrix}.$$

Recall that given a sequence of numbers/moments $\mathbf{m} := (m_n)_{n=0}^{\infty}$, let $P_n(y)$ be the monic orthogonal polynomials with respect to m_n. Namely, for all $0 \leq r < n$

$$y^r P_n(y)\Big|_{y^k = m_k} = 0.$$

Suppose P_n satisfies the three-term recurrence:

$$P_{n+1}(y) = (y - s_n)P_n(y) - t_n P_{n-1}(y).$$

Theorem. (1) If $m_{2k+1} = 0$ for all $k \in \mathbb{N}$, then, $s_n = 0$;
Recall The monic orthogonal polynomials with respect to E_n, $Q_n(y)$, satisfy

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And

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$$\Delta_n(\mathbf{m}) = m_0^{n+1}(-t_1)^n(-t_2)^{n-1}\cdots(-t_{n-1})^2(-t_n),$$

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2nd Task

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$$\left(\frac{t}{e^t-1}\right)^p e^{xt} = \sum_{n=0}^{\infty} B_n^{(p)}(x)\frac{t^n}{n!} \quad \text{and} \quad \left(\frac{2}{e^z+1}\right)^p e^{xz} = \sum_{n=0}^{\infty} E_n^{(p)}(x)\frac{z^n}{n!},$$

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Then, with the lemma for shifted and scaled random variables, we see

$$\varrho_{n+1}^{(p)}(y) = \left(y - x + \frac{p}{2}\right) \varrho_n^{(p)}(y) + b_n^{(p)} \varrho_{n-1}^{(p)}(y)$$

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Conjecture on $b_n^{(p)}$

The first several terms of $b_n^{(p)}$ is given in the following table

	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$
$n=1$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{5}{12}$
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$n=3$	$\frac{81}{140}$	$\frac{372}{455}$	$\frac{1339}{1260}$	$\frac{2109}{1610}$	$\frac{1527}{980}$
$n=4$	$\frac{64}{63}$	$\frac{3736}{2821}$	$\frac{138688}{84357}$	$\frac{668543}{339549}$	$\frac{171830}{74823}$
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- The first row is $p/12$
- The second row is $(5p+3)/30$

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Conjecture. [K. Dilcher]

$$b_3^{(p)} = \frac{175p^2 + 315p + 158}{140(2p+3)};$$

$$b_4^{(p)} = \frac{6125p^4 + 25725p^3 + 41965p^2 + 29547p + 7230}{21(5p+3)(175p^2 + 315p + 158)};$$

$$b_5^{(p)} = 25(5p+3)(471625p^6 + 3678675p^5 + 12324235p^4 + 22096305p^3 + 22009540p^2 + 11549748p + 2519472) \Big/ (132(175p^2 + 315p + 158)(6125p^4 + 25725p^3 + 41965p^2 + 29547p + 7230)).$$

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random variable	moments	monic orthogoal polynomial
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CX	$C^n m_n$	$\tilde{P}_n(y): \quad \tilde{P}_{n+1}(y) = (y - Cs_n)\tilde{P}_n(y) - C^2 t_n \tilde{P}_{n-1}(y)$
$X + Y$	Convolution	???

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- **Theorem**. [L. Jiu and D. Shi]

$$\Omega_{n+1}^{(p)}(y) = \left(y - x + \frac{p}{2}\right)\Omega_n^{(p)}(y) + \frac{n(n+p-1)}{4}\Omega_{n-1}^{(p)}(y).$$

Meixner-Pollaczek polynomials

The *Meixner-Pollaczek polynomials* are defined by

$$P_n^{(\lambda)}(y;\phi) := \frac{(2\lambda)_n}{n!} e^{in\phi} \, {}_2F_1\left(\left. \begin{matrix} -n, \lambda + iy \\ 2\lambda \end{matrix} \right| 1 - e^{-2i\phi} \right),$$

where $(x)_n := x(x+1)(x+2)\cdots(x+n-1)$ is the Pochhammer symbol and ${}_2F_1$ is the hypergeometric function

$${}_pF_q\left(\left. \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \right| t \right) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \cdot \frac{t^n}{n!}.$$

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The key property for Meixner-Pollaczek polynomials

$$P_n^{(\lambda+\mu)}(y_1 + y_2, \phi) = \sum_{k=0}^{n} P_k^{(\lambda)}(y_1, \phi) \, P_{n-k}^{(\mu)}(y_2, \phi)$$

does not hold for continuous Hahn polynomials.

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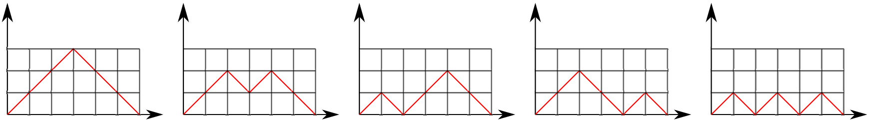

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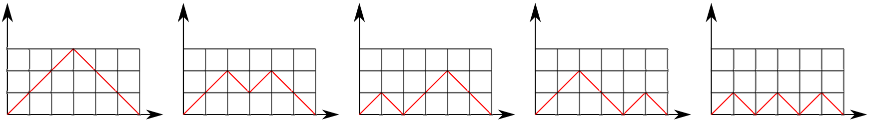

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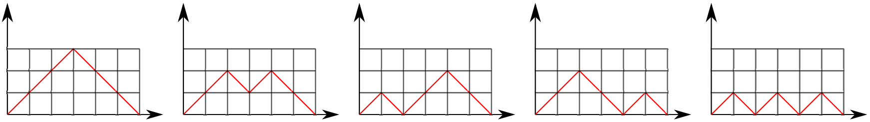

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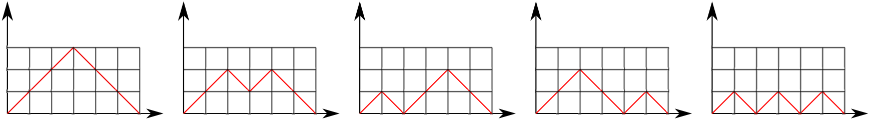

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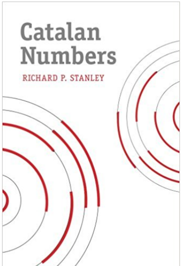

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$$R_{n+1}(y) = \left(y + \frac{1}{2}\right) R_n(y) + \frac{n^4}{4(2n+1)(2n-1)} R_{n-1}(y)$$

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$$P(n;y) = \sum_{k=1}^{n-1} \alpha_{n,k} R_k(y).$$

For some constants $\alpha_{n,k}$ that are independent of y.

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Proof. By induction on the degree of P.

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- $P(2;y) = (y+x+1)^2 - (y+x)^2 - 2x$

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- $P(2; y) = (y + x + 1)^2 - (y + x)^2 - 2x = 2y + 1 = 2R_1;$
- $P(3; y) = 3y^2 + (3 + 6x)y + 3x + 1 = 3\left(y^2 + y + \frac{1}{3}\right) + 6xy + 3x$

What's Next?

Exmaple.

$$\begin{aligned} P(n;y) &= (y+x+1)^n - (y+x)^n - nx^{n-1} \\ R_1 &= y + \frac{1}{2} \\ R_2 &= y^2 + y + \frac{1}{3} \\ R_3 &= y^3 + \frac{3}{2}y^2 + \frac{11}{10}y + \frac{3}{10} \end{aligned}$$

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- $P(4;y) = 4R_3 + 12xR_2 + (12x^2 - \frac{2}{5})R_1.$

What's Next?

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Define

$$\frac{e^{xt}}{{}_1F_1\left(\left.\begin{matrix}a\\a+b\end{matrix}\right| t\right)} = \sum_{n=0}^{\infty} B_n^{(a,b)}(x) \frac{t^n}{n!}.$$

End

Thank you!

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