Random Walk & Identities

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Dalhousie University Number Theory Seminar

February 25th, 2019

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Pledge Your Allegiance
Subway Series

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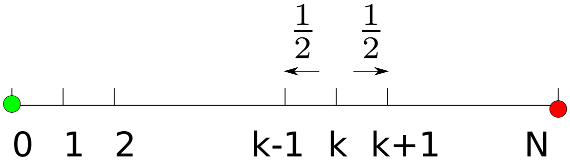

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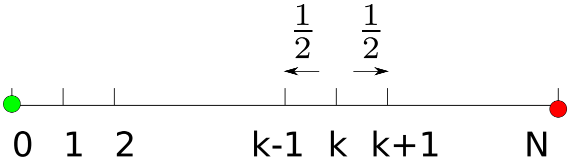

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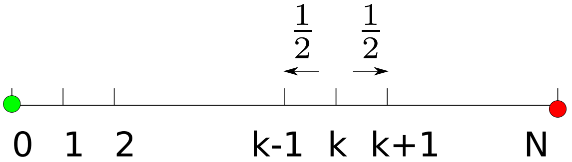

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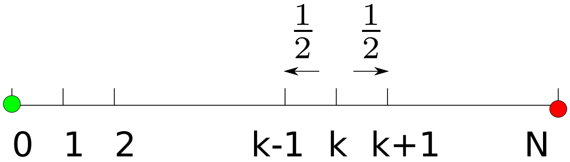

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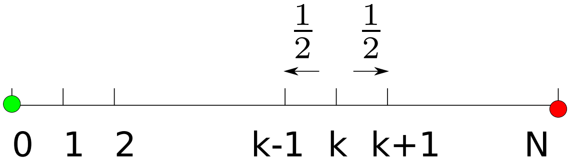

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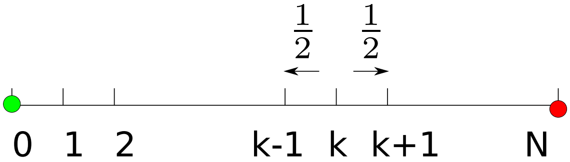

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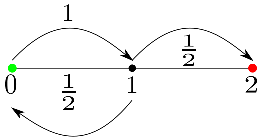

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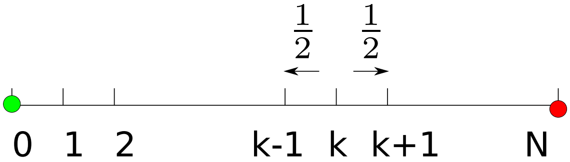

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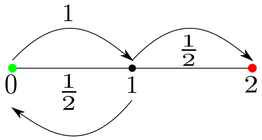

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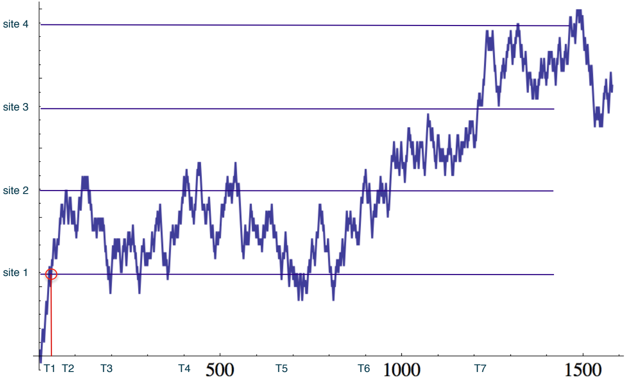

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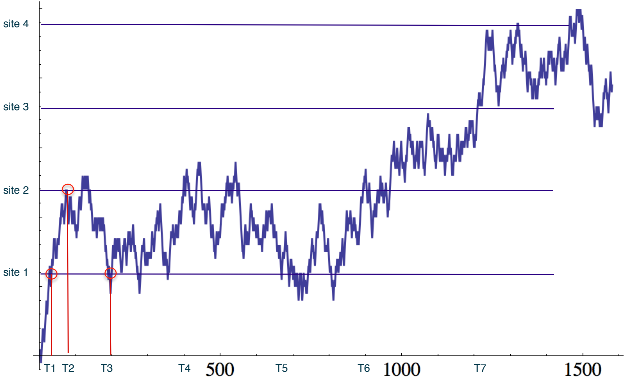

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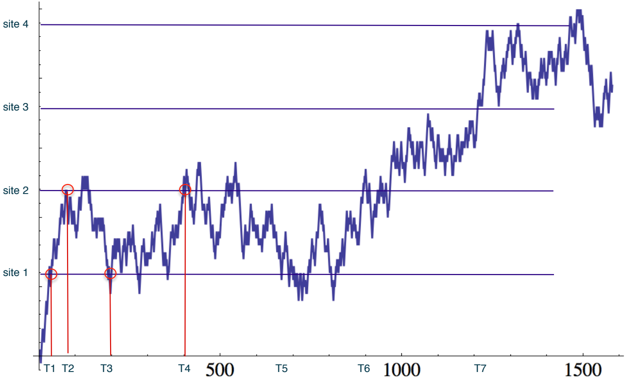

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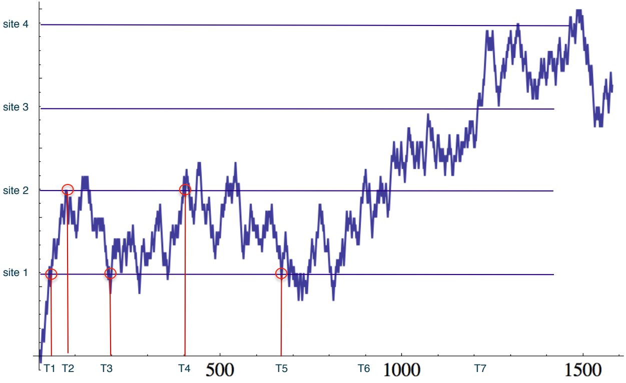

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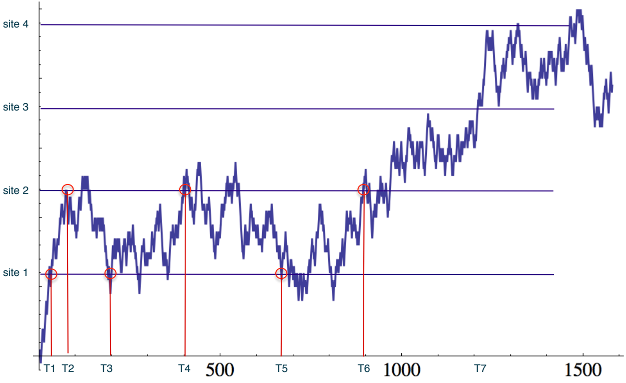

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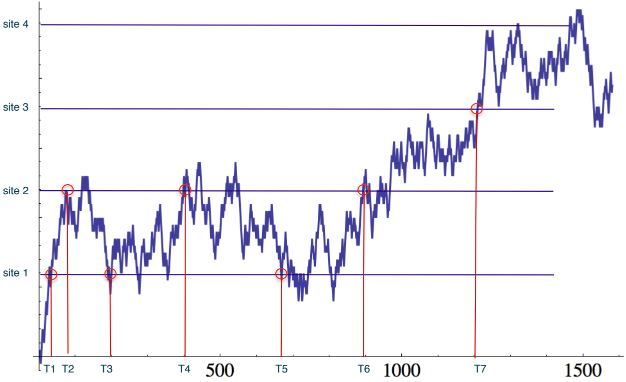

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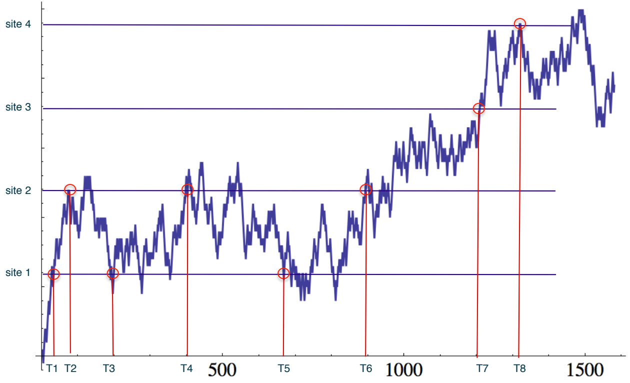

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[Klebanov2012]L. B. Klebanov, A. V. Kakosyan, S. T. Rachev, and G. Temnov. On a class of distributions stable under random summation. *J. Appl. Prob.*, 49:303–318, 2012.

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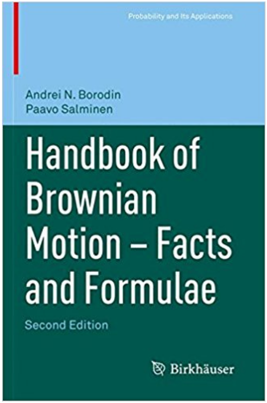
Taking moments:

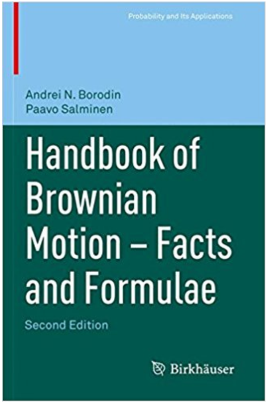
- LHS: $\mathbb{E}\left[\left(x + iL - \frac{1}{2} \right)^n \right] = E_n(x)$;
- RHS: Each $\nu_N = \ell$, with probability $p_\ell^{(N)}$ and
  $\mathbb{E}\left[\left(i \sum_{j=1}^{\ell} L_j - \frac{\ell}{2} + Nx - \frac{N}{2} + \frac{\ell}{2} \right)^n \right] = E_n^{(\ell)} \left(\frac{\ell - N}{2} + Nx \right).$ □

Hitting Time



Hitting Time

- Reflected (Reflecting) Brownian Motion in $\mathbb{R}_+$: W_t =distance to 0 at time t.



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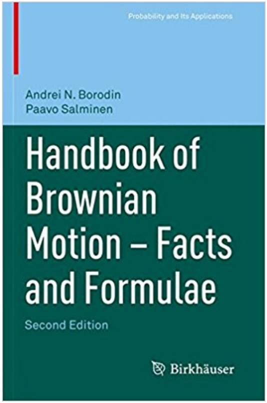

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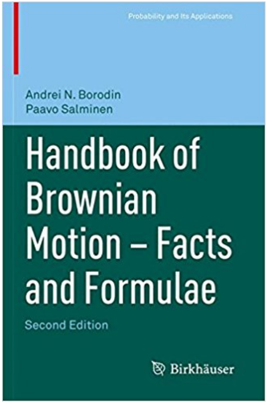

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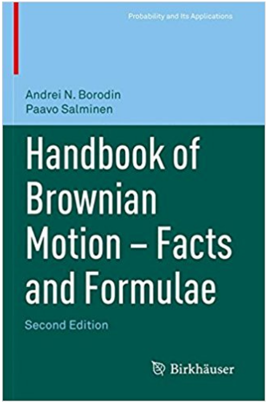

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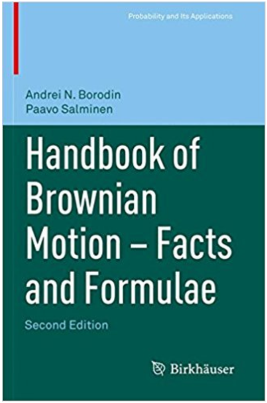

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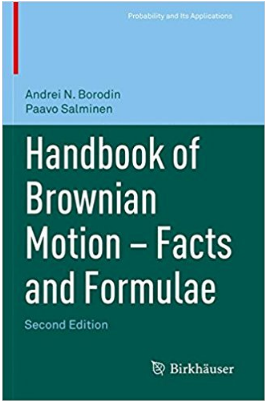

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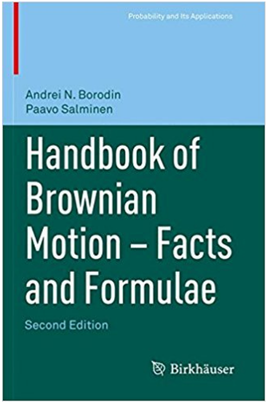

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$$\frac{2}{1+e^s} e^{sx} = \sum_{n=0}^{\infty} E_n(x) \frac{s^n}{n!}$$

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Now write

$$T = (T - T_{\ell-1}) + (T_{\ell-1} - T_{\ell-2}) + \cdots + (T_1 - 0)$$

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$$Z_N = \frac{1}{N} \sum_{j=1}^{\nu_N} L_j \sim L.$$

My Goal

$$E_n(x) = \frac{1}{N^n} \sum_{\ell=N}^{\infty} p_\ell^{(N)} E_n^{(\ell)} \left(\frac{\ell - N}{2} + Nx \right) \Longleftrightarrow \frac{1}{N} \sum_{j=1}^{\nu_N} L_j \sim L \sim \operatorname{sech}(x)$$

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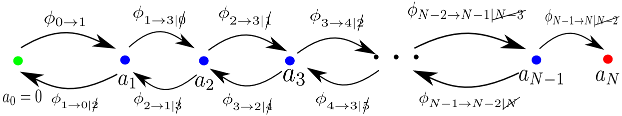

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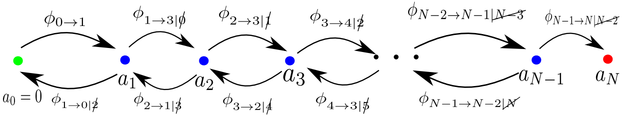

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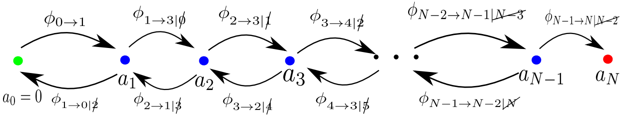

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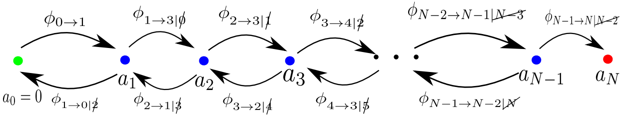

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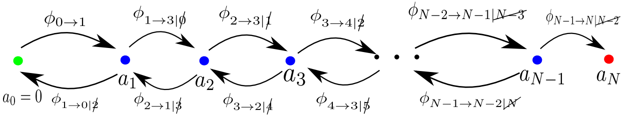

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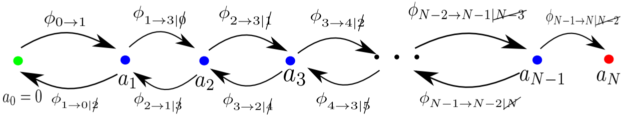

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If anyone has an idea, please let me know.

1-dim, 1-loop

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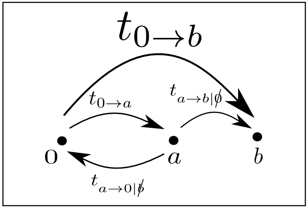

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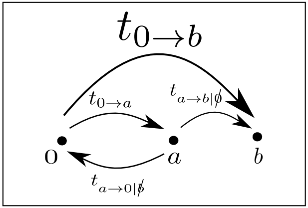

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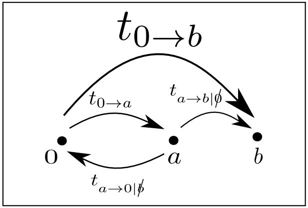

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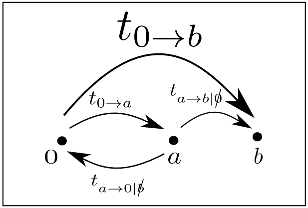

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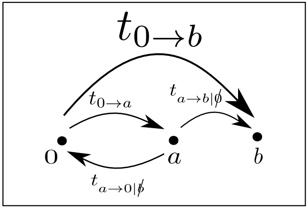

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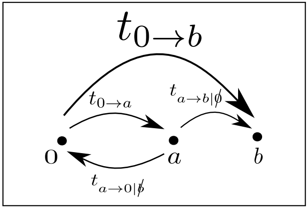

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Proposition. [LJ. and C. Vignat]

$$E_n\left(\frac{x}{2b}+\frac{3}{2}-2\frac{a}{b}\right)-E_n\left(\frac{x}{b}+\frac{1}{2}\right)=\frac{(n+1)\left(1-2\frac{a}{b}\right)2^n a^n}{b^n}\sum_{\ell=0}^{\infty}\frac{\frac{a}{b}\left(1-\frac{a}{b}\right)^{\ell}}{}B_n^{(\ell+1)}\left(\frac{x+b}{4a}+\frac{\ell}{2}\right).$$

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How about 2-loops?

1-dim, 1-loop

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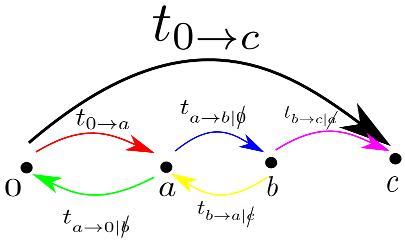
$$E_n\left(\frac{x}{2b}+\frac{3}{2}-2\frac{a}{b}\right)-E_n\left(\frac{x}{b}+\frac{1}{2}\right)=\frac{(n+1)\left(1-2\frac{a}{b}\right)2^n a^n}{b^n}\sum_{\ell=0}^{\infty}\underline{\frac{a}{b}\left(1-\frac{a}{b}\right)^{\ell}}B_n^{(\ell+1)}\left(\frac{x+b}{4a}+\frac{\ell}{2}\right).$$

- $\frac{a}{b}\left(1-\frac{a}{b}\right)^{\ell}$ are the probability weights of a geometric distribution with parameter a/b.

The case $b=2a$, i.e., equally distributed sites, gives

$$0=0.$$

How about 2-loops?



1-dim, 1-loop

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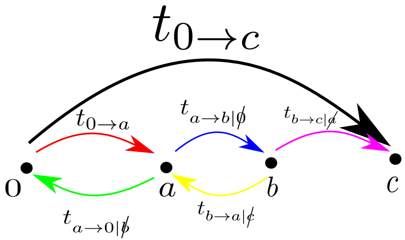
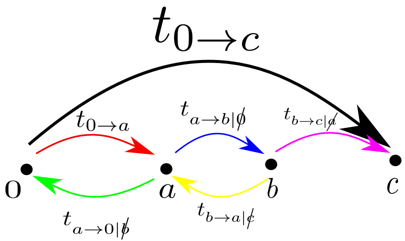


t, t, t, t, t, t

1-dim, 2-loops



$t_{0\to c}$
$t_{0\to a}$
$t_{a\to b|\not 0}$
$t_{b\to c|\not a}$
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o
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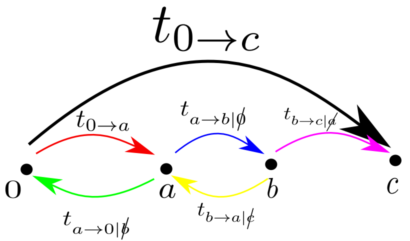
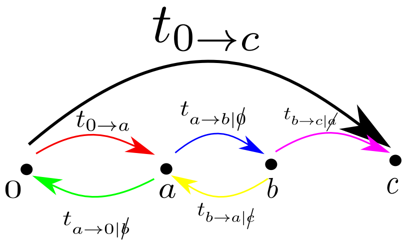
b
c

1-dim, 2-loops



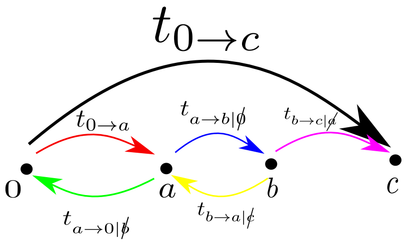
$t \;=\; t$

1-dim, 2-loops



$$t \quad = \quad t + t$$

1-dim, 2-loops



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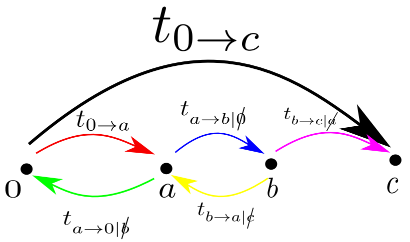
1-dim, 2-loops



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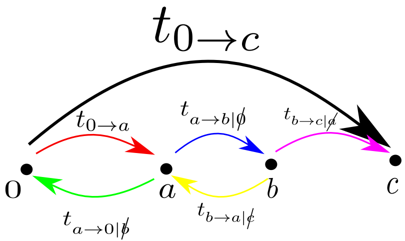
1-dim, 2-loops



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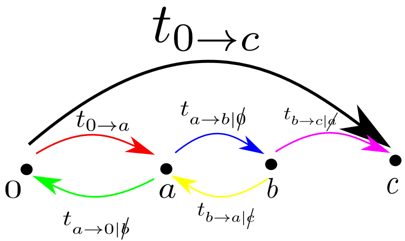
1-dim, 2-loops



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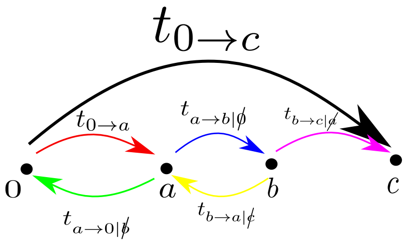
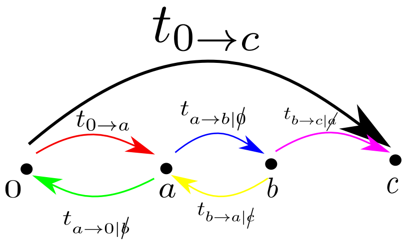
1-dim, 2-loops



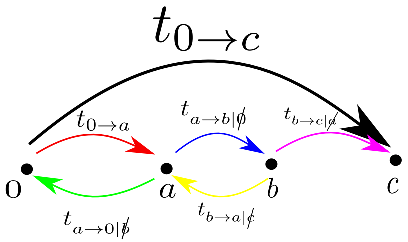
$$t = t + t + t + t + t + t + \cdots + t$$

1-dim, 2-loops



$$
\begin{aligned}
t &= t + t + t + t + t + t + \cdots + t \\
&= t + t + t + \underbrace{(t + t) + \cdots + (t + t)}_{k \text{ loops}} + \underbrace{(t + t) + \cdots + (t + t)}_{\ell \text{ loops}}
\end{aligned}
$$

1-dim, 2-loops



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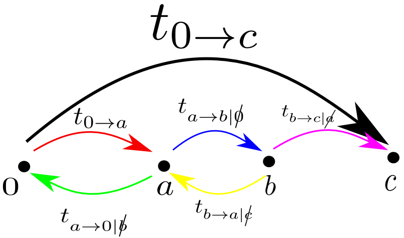
We can generalize it to n-loop model.

1-dim, 2-loops



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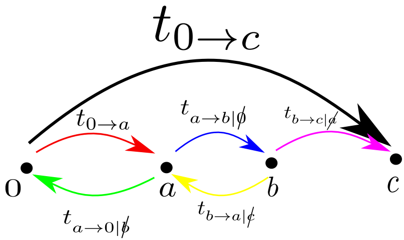
\end{aligned}
$$

We can generalize it to n-loop model.

Unfortunately, this is WRONG.....

1-dim, 2-loops



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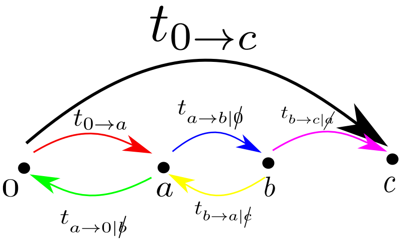
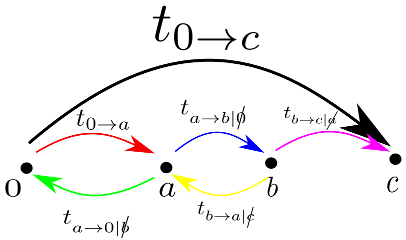
1-dim, 2-loops



$$\phi = \phi \cdot \phi \cdot \phi \cdot \left[\sum_{k=0}^{\infty} (\phi\phi)^k\right] \left[\sum_{\ell=0}^{\infty} (\phi\phi)^\ell\right] = \frac{\phi \cdot \phi \cdot \phi}{(1-\phi\phi)(1-\phi\phi)}$$

1-dim, 2-loops



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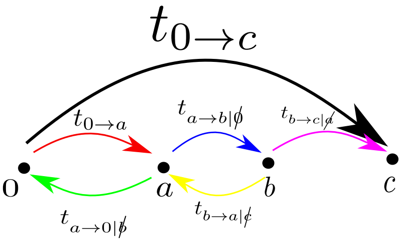
1-dim, 2-loops



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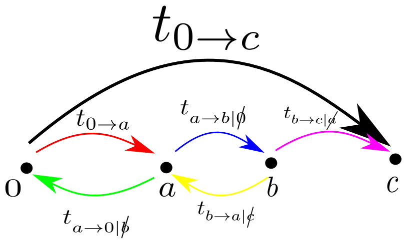
1-dim, 2-loops



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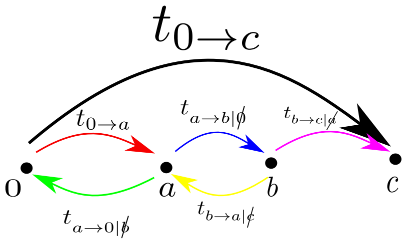
1-dim, 2-loops



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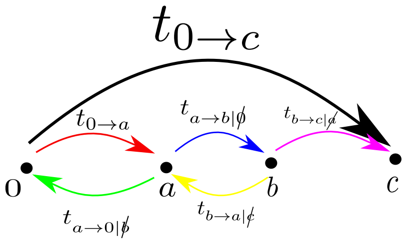
1-dim, 2-loops



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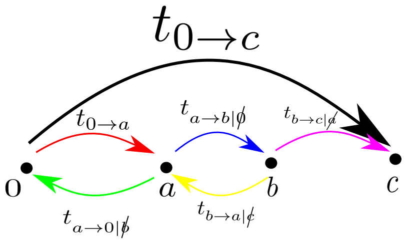
1-dim, 2-loops



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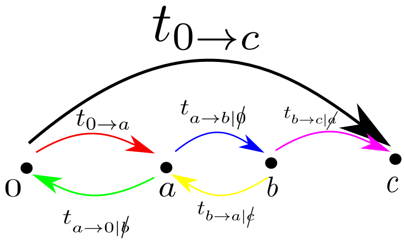
$$= \frac{2\cosh w}{(2\cosh^2 w - 1)\,(4\cosh^2 w - 1)}$$

1-dim, 2-loops



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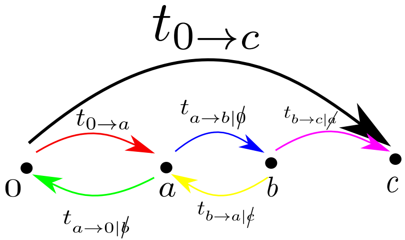
$$= \frac{2\cosh w}{(2\cosh^2 w-1)\,(4\cosh^2 w-1)} \neq \frac{1}{\cosh(3w)} = \frac{1}{4\cosh^3 w - 3\cosh w}$$

Problem



$t_{0\to c}$
$t_{0\to a}$
$t_{a\to b|\not 0}$
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0
a
b
c
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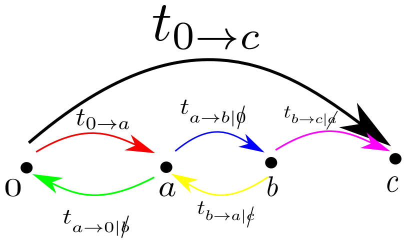
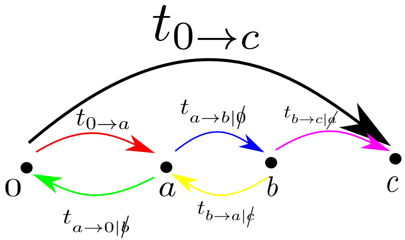
Problem



$$\phi = \phi \cdot \phi \cdot \phi \cdot \text{[Loops]}$$

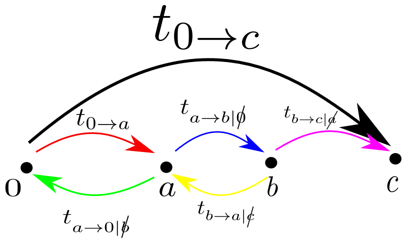
Problem



$$\phi = \phi \cdot \phi \cdot \phi \cdot [\text{Loops}]$$

$$\frac{1}{\cosh(3w)} = \frac{1}{\cosh(w)} \cdot \frac{\sinh(w)}{\sinh(2w)} \cdot \frac{\sinh(w)}{\sinh(2w)} \cdot [\text{Loops}]$$

Problem



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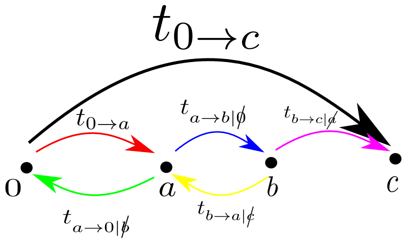
Problem



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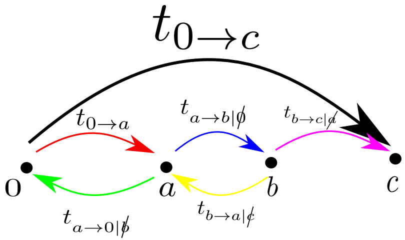
$$[\text{Loops}] = \frac{4\cosh^2 w}{4\cosh^3 w - 3\cosh w}$$

Problem

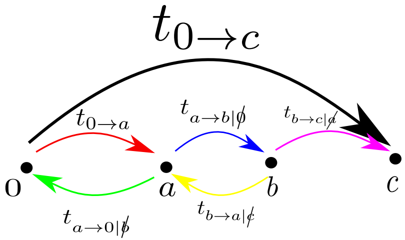


$$\phi = {\color{red}\phi} \cdot {\color{blue}\phi} \cdot {\color{magenta}\phi} \cdot [\text{Loops}]$$

$$\frac{1}{\cosh(3w)} = {\color{red}\frac{1}{\cosh(w)}} \cdot {\color{blue}\frac{\sinh(w)}{\sinh(2w)}} \cdot {\color{magenta}\frac{\sinh(w)}{\sinh(2w)}} \cdot [\text{Loops}] = \frac{1}{4\cosh^2 w} \cdot [\text{Loops}]$$

$$[\text{Loops}] = \frac{4\cosh^2 w}{4\cosh^3 w - 3\cosh w} = \frac{1}{1 - \frac{3}{4\cosh^2 w}} = \sum_{\ell=0}^{\infty} \left(\frac{3}{4\cosh^2 w} \right)^{\ell}.$$

Problem



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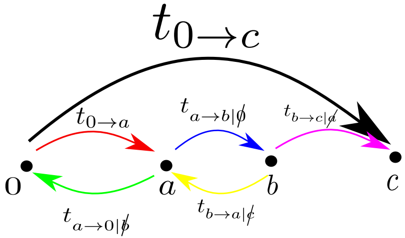
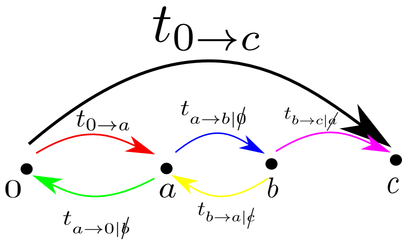
$$\phi\phi + \phi\phi = \frac{1}{\cosh(w)} \frac{\sinh(w)}{\sinh(2w)} + \frac{\sinh(w)}{\sinh(2w)} \frac{\sinh(w)}{\sinh(2w)} = \frac{3}{4\cosh^2 w}$$

Explanation

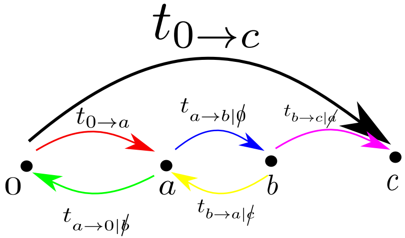


Explanation



$$t = t + t + t + \underbrace{(t+t) + \cdots + (t+t)}_{k \text{ loops}} + \underbrace{(t+t) + \cdots + (t+t)}_{\ell \text{ loops}}$$

Explanation

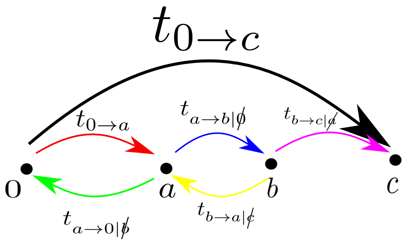


$$t = t + t + t + \underbrace{(t+t) + \cdots + (t+t)}_{k \text{ loops}} + \underbrace{(t+t) + \cdots + (t+t)}_{\ell \text{ loops}}$$

For instance,

$$t = \underbrace{(t+t) + \cdots + (t+t)}_{k_1 \text{ loops}} + t + \underbrace{(t+t) + \cdots + (t+t)}_{\ell \text{ loops}} + \underbrace{(t+t) + \cdots + (t+t)}_{k_2 \text{ loops}} + t + t$$

Explanation



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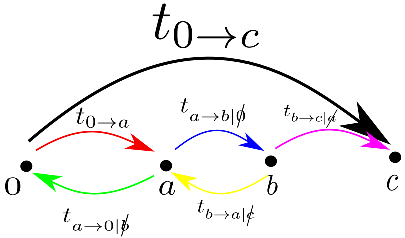
Let both k_1 and $k_2 \to \infty$.

Explanation



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Let both k_1 and $k_2 \to \infty$.

$$\phi\phi + \phi\phi = \frac{1}{\cosh(w)} \frac{\sinh(w)}{\sinh(2w)} + \frac{\sinh(w)}{\sinh(2w)} \frac{\sinh(w)}{\sinh(2w)} = \frac{3}{4\cosh^2 w}$$

Two-loops

Two-loops

$$I := \phi_{a\to b|\not A}\,\phi_{b\to a|\not c}, \qquad II := \phi_{b\to c|\not a}\,\phi_{c\to b|\not B}$$

- k loops of I followd by l loops of II, with $k, l = 0, 1, \ldots,$ which gives

$$\sum_{k.l} I^k II^l = \frac{1}{1-I} \cdot \frac{1}{1-II};$$

- k_1 loops of I followed by l_1 loops of II, then followed by k_2 loops of I and finally followed by l_2 loops of II, with k_1, l_2 nonnegative and k_2, l_1 positive, which gives

$$\sum_{k_1.l_2=0,k_2,l_1=1}^{\infty} I^{k_1} II^{l_1} I^{k_2} II^{l_2} = \frac{I \cdot II}{(1-I)^2\,(1-II)^2};$$

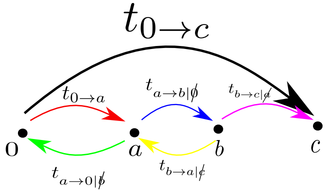
- the general term will be k_1 loops of $I \to l_1$ loops of $II \to \cdots \to k_n$ loops of $I \to l_n$ loops of II, with k_1, l_n nonnegtive and the rest indices positive, which gives

$$\frac{(I \cdot II)^{n-1}}{(1-I)^n\,(1-II)^n}.$$

Therefore, loops I and II contribute as

$$\sum_{n=1}^{\infty} \frac{(I \cdot II)^{n-1}}{(1-I)^n\,(1-II)^n} = \frac{1}{1-(I+II)} = \sum_{k=0}^{\infty} (I+II)^k .$$

Two Loops



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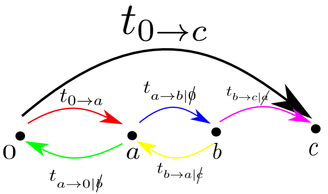

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Proposition. [LJ. and C. Vignat] For any positive integer n,

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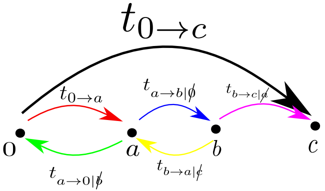

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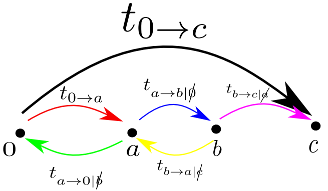

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$$S_\nu(x,y) := (xy)^{-\nu}\left[I_\nu(x)K_\nu(y) - K_\nu(x)I_\nu(y)\right],$$

and

$$I_\nu(x) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!\Gamma(\ell+\nu+1)}\left(\frac{x}{2}\right)^{2\ell+\nu}, \quad K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\nu\pi)}.$$

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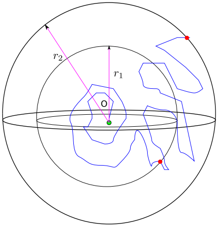

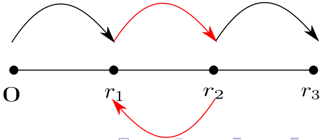

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$$\sum_{n=0}^{\infty}E_n^{(p)}(x)\frac{z^n}{n!}=\left(\frac{2}{e^z+1}\right)^{p}e^{xz}\Rightarrow E_1^{(2k+2)}(x)=x-(k+1).$$

$n = 3 \Leftrightarrow \nu = 1/2$

Proposition. [LJ. and C. Vignat]

$$\frac{3^{n+1}}{n+1}\left[B_{n+1}\left(\frac{x}{6}+\frac{5}{6}\right)-B_{n+1}\left(\frac{x}{6}+\frac{1}{2}\right)\right]=\sum_{k\geq 0}\frac{3}{4}\left(\frac{1}{4}\right)^k E_n^{(2k+2)}\left(\frac{x+3+2k}{2}\right). \quad (\circledast)$$

Corollary 1.

$$B_{2m}=\frac{m}{\left(1-2^{1-2m}\right)\left(3^{2m}-1\right)}\sum_{k\geq 0}\left(\frac{1}{4}\right)^k E_{2m-1}^{(2k+2)}\left(k+\frac{3}{2}\right).$$

Corollary 2. Take $n=1$ in $(\circledast)$.

$$B_2(x)=x^2-x+\frac{1}{6}\Rightarrow \text{LHS}=\frac{x+1}{2},$$

$$\sum_{n=0}^{\infty}E_n^{(p)}(x)\frac{z^n}{n!}=\left(\frac{2}{e^z+1}\right)^p e^{xz}\Rightarrow E_1^{(2k+2)}(x)=x-(k+1).$$

$$\sum_{k\geq 0}\frac{3}{4}\left(\frac{1}{4}\right)^k\left(\frac{x+3+2k}{2}-k-1\right)=\sum_{k\geq 0}\frac{3}{4}\left(\frac{1}{4}\right)^k\left(\frac{x+1}{2}\right)=\frac{x+1}{2}.$$

Proposition. [LJ. and C. Vignat] For any positive integer n,

$$3^n B_n\left(\frac{x+4}{6}\right) = \sum_{k=0}^{\infty} \frac{1}{2^k} E_n^{(2k+2)}\left(\frac{x+2k+3}{2}\right).$$

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Several remarks are in order at this point:

- the identities obtained from this approach are not of the usual, convolutional type. Rather, they are connection-type identities between the usual Bernoulli and Euler polynomials and their higher-order counterparts;
- these inherently involve a mixture of higher-order Bernoulli and Euler polynomials;
- the interest of this approach is that each term in such a decomposition can be related to a physical object, namely one loop in a trajectory of a random process;
- this work should be considered as only a first approach to a more general project in which the richness of the possible setups for random walks is expected to generate a number of non-trivial identities about more general special functions.

End

Thank you!

Connection Coefficients for Higher-order Bernoulli and Euler Polynomials:

A Random Walk Approach

https://arxiv.org/abs/1809.04636