Random Walk Approaches to Identities on Higher-order Bernoulli and Euler Polynomials

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Bernoulli and Euler number, polynomials.

- Bernoulli numbers B_n:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!};$$

- Euler numbers E_n:

$$\frac{2e^t}{e^{2t} + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!};$$

- Bernoulli polynomial $B_n(x)$:

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!};$$

- Euler polynomial $E_n(x)$:

$$\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!};$$

- Bernoulli polynomial of order p $B_n^{(p)}(x)$:

$$\left(\frac{t}{e^t - 1}\right)^p e^{xt} = \sum_{n=0}^{\infty} B_n^{(p)}(x) \frac{t^n}{n!};$$

- Euler polynomial of order p $E_n^{(p)}(x)$:

$$\left(\frac{2}{e^z + 1}\right)^p e^{xz} = \sum_{n=0}^{\infty} E_n^{(p)}(x) \frac{z^n}{n!};$$

$$B_n^{(1)}(x) = B_n(x);\ B_n(0) = B_n;\ E_n^{(1)}(x) = E_n(x);\ E_n(1/2) = E_n/2^n.$$

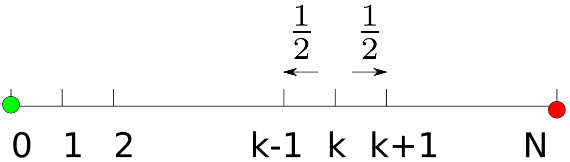
Theorem (LJ, V. H. Moll, and C. Vignat). $\forall N \in \mathbb{Z}_+$

$$E_n(x) = \frac{1}{N^n} \sum_{\ell=N}^{\infty} p_\ell^{(N)} E_n^{(\ell)} \left(\frac{\ell - N}{2} + Nx \right),$$

where

$$\frac{1}{T_N\left(\frac{1}{z}\right)} = \sum_{\ell=0}^{\infty} p_\ell^{(N)} z^\ell, \qquad T_N(\cos\theta) = \cos(N\theta).$$

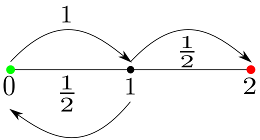
Random Walk



- 0 is the source and N is the sink;
- at each $k = 1, \dots, N-1$, it is a "fair coin" walk;
- let ν_N be the random number of steps for this process.

$$p_\ell^{(N)} = \mathbb{P}(\nu_N = \ell)$$

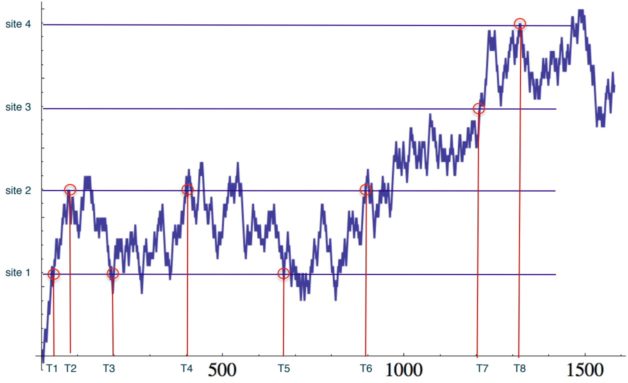
Example. $\underline{N = 2:}$



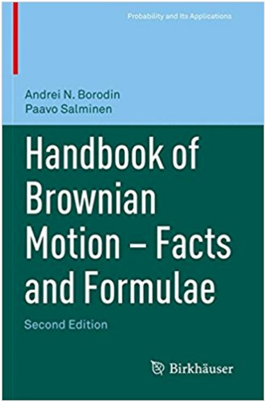
$$p_\ell^{(2)} = \begin{cases} \frac{1}{2^k}, & \text{if } \ell = 2k; \\ 0, & \text{otherwise} \end{cases}$$

$$\cos(2\theta) = 2\cos^2\theta - 1 \Rightarrow T_2(z) = 2z^2 - 1 \Rightarrow \frac{1}{T_2(1/z)} = \frac{z^2}{2 - z^2} = \sum_{k=1}^{\infty} \frac{z^{2k}}{2^k}.$$

Reflected Brownian Motion



Random Walk and Hitting Time



- Reflected (Reflecting) Brownian Motion in $\mathbb{R}_+$: W_t =distance to 0 at time t.
- Hitting times: $H_z := \min_t \{W_t = z\}$.
- $$\mathbb{E}_x\left[e^{-\alpha H_z}\right] = \begin{cases} \frac{\cosh(xw)}{\cosh(zw)}, & 0 \le x \le z; \\ e^{-(x-z)w}, & z \le x. \end{cases}$$

  1. $w = \sqrt{2\alpha}$;
  2. $\mathbb{E}_x$ means it starts with point x (instead of 0);
  3. $$\frac{2}{1+e^s} e^{sx} = \sum_{n=0}^{\infty} E_n(x) \frac{s^n}{n!} = \frac{e^{s\left(x-\frac{1}{2}\right)}}{\cosh\left(\frac{s}{2}\right)}.$$

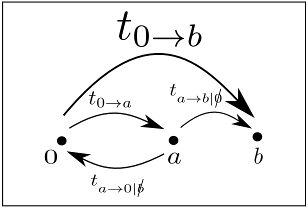
1-dim, 1-loop

With $p \le q \le r$, $w = \sqrt{2\alpha}$

$$\phi_{p\to q} := \mathbb{E}_p\left[e^{-\alpha H_q}\right] = \frac{\cosh(pw)}{\cosh(qw)},$$

$$\phi_{q\to p|\not r} := \mathbb{E}_q\left[e^{-\alpha H_p} | W_t < r\right] = \frac{\sinh((r-q)w)}{\sinh((r-p)w)},$$

$$\phi_{q\to r|\not p} := \mathbb{E}_q\left[e^{-\alpha H_r} | W_t > p\right] = \frac{\sinh((q-p)w)}{\sinh((r-p)w)},$$



- The hitting time $t_{0\to b}$ can be decomposed as

$$t_{0\to b} = \underbrace{\left(t_{0\to a} + t_{a\to 0|\not b}\right) + \cdots + \left(t_{0\to a} + t_{a\to 0|\not b}\right)}_{\ell \text{ copies}} + t_{0\to a} + t_{a\to b|\not 0}$$

- Generating functions:

$$\phi_{0\to b} = \phi_{0\to a}\phi_{a\to b|\not 0} \sum_{\ell=0}^{\infty} \left(\phi_{0\to a}\phi_{a\to 0|\not b}\right)^{\ell}$$

$$\phi_{0\to b} = \frac{1}{\cosh(bw)},$$

$$\text{RHS} = \frac{1}{\cosh(aw)} \cdot \frac{\sinh(aw)}{\sinh(bw)} \sum_{\ell=0}^{\infty} \left[\frac{1}{\cosh(aw)} \cdot \frac{\sinh((b-a)w)}{\sinh(bw)}\right]^{\ell}$$

$$= \frac{1}{\cosh(aw)} \cdot \frac{\sinh(aw)}{\sinh(bw)} \cdot \frac{1}{1 - \frac{1}{\cosh(aw)} \cdot \frac{\sinh((b-a)w)}{\sinh(bw)}}$$

1-dim, 1-loop

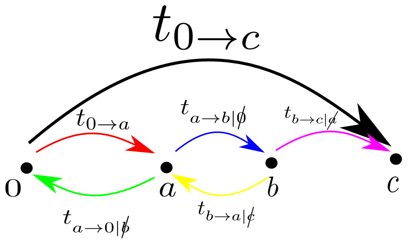
Proposition. [LJ. and C. Vignat]

$$E_n\left(\frac{x}{2b}+\frac{3}{2}-2\frac{a}{b}\right)-E_n\left(\frac{x}{b}+\frac{1}{2}\right)=\frac{(n+1)\left(1-2\frac{a}{b}\right)2^n a^n}{b^n}\sum_{\ell=0}^{\infty}\underline{\frac{a}{b}\left(1-\frac{a}{b}\right)^{\ell}}B_n^{(\ell+1)}\left(\frac{x+b}{4a}+\frac{\ell}{2}\right).$$

- $\frac{a}{b}\left(1-\frac{a}{b}\right)^{\ell}$ are the probability weights of a geometric distribution with parameter a/b.
- The case $b=2a$, i.e., equally distributed sites, gives

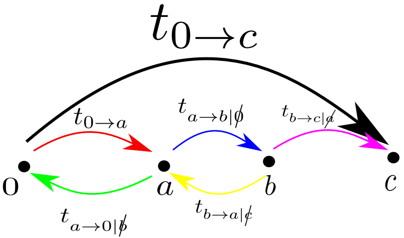
$$0=0.$$

How about 2-loops?



$t,\ t,\ t,\ t,\ t,\ t$

1-dim, 2-loops

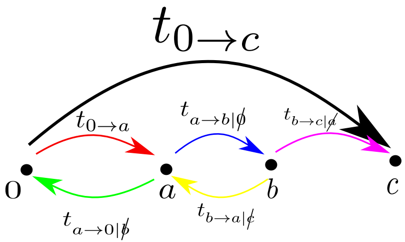


$$\begin{aligned} t &= t + t + t + t + t + t + \cdots + t + t \\ &= t + t + t + \underbrace{(t + t) + \cdots + (t + t)}_{k \text{ loops}} + \underbrace{(t + t) + \cdots + (t + t)}_{\ell \text{ loops}} \end{aligned}$$

We can generalize it to n-loop model.

Unfortunately, this is WRONG.....

1-dim, 2-loops



Suppose it is true. Then,

$$\phi = \phi\cdot\phi\cdot\phi\cdot\left[\sum_{k=0}^{\infty}(\phi\phi)^k\right]\left[\sum_{\ell=0}^{\infty}(\phi\phi)^\ell\right] = \frac{\phi\cdot\phi\cdot\phi}{(1-\phi\phi)(1-\phi\phi)}$$

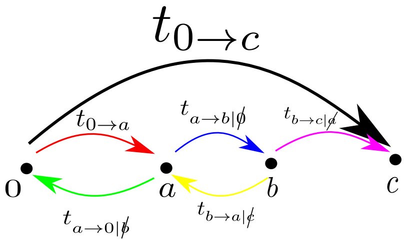
Let $a = 1$, $b = 2$ and $c = 3$.

$$\text{LHS} = \phi = \phi_{0\to 3} = 1/\cosh(3w)$$

$$\text{RHS} = \frac{\frac{1}{\cosh(w)}\cdot\frac{\sinh(w)}{\sinh(2w)}\cdot\frac{\sinh(w)}{\sinh(2w)}}{\left(1-\frac{1}{\cosh(w)}\frac{\sinh(w)}{\sinh(2w)}\right)\left(1-\frac{\sinh(w)}{\sinh(2w)}\frac{\sinh(w)}{\sinh(2w)}\right)} = \frac{\frac{1}{4\cosh^3 w}}{\left(1-\frac{1}{2\cosh^2 w}\right)\left(1-\frac{1}{4\cosh^2 w}\right)}$$

$$= \frac{2\cosh w}{(2\cosh^2 w - 1)(4\cosh^2 w - 1)} \neq \frac{1}{\cosh(3w)} = \frac{1}{4\cosh^3 w - 3\cosh w}$$

Problem

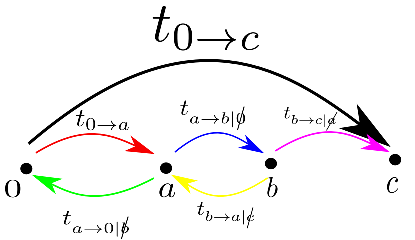


$$\phi = \color{red}{\phi} \cdot \color{blue}{\phi} \cdot \color{magenta}{\phi} \cdot [\text{Loops}]$$

$$\frac{1}{\cosh(3w)} = \color{red}{\frac{1}{\cosh(w)}} \cdot \color{blue}{\frac{\sinh(w)}{\sinh(2w)}} \cdot \color{magenta}{\frac{\sinh(w)}{\sinh(2w)}} \cdot [\text{Loops}] = \frac{1}{4\cosh^2 w} \cdot [\text{Loops}]$$

$$[\text{Loops}] = \frac{4\cosh^2 w}{4\cosh^3 w - 3\cosh w}$$

Problem



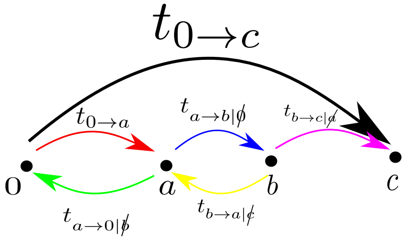
$$\phi = \phi \cdot \phi \cdot \phi \cdot [\text{Loops}]$$

$$\frac{1}{\cosh(3w)} = \frac{1}{\cosh(w)} \cdot \frac{\sinh(w)}{\sinh(2w)} \cdot \frac{\sinh(w)}{\sinh(2w)} \cdot [\text{Loops}] = \frac{1}{4\cosh^2 w} \cdot [\text{Loops}]$$

$$[\text{Loops}] = \frac{4\cosh^2 w}{4\cosh^3 w - 3\cosh w} = \frac{1}{1 - \frac{3}{4\cosh^2 w}} = \sum_{\ell=0}^{\infty} \left(\frac{3}{4\cosh^2 w}\right)^{\ell}.$$

$$\phi\phi + \phi\phi = \frac{1}{\cosh(w)} \frac{\sinh(w)}{\sinh(2w)} + \frac{\sinh(w)}{\sinh(2w)} \frac{\sinh(w)}{\sinh(2w)} = \frac{3}{4\cosh^2 w}$$

Explanation



$$t = t + t + t + \underbrace{(t + t) + \cdots + (t + t)}_{k \text{ loops}} + \underbrace{(t + t) + \cdots + (t + t)}_{\ell \text{ loops}}$$

For instance,

$$t = \underbrace{(t + t) + \cdots + (t + t)}_{k_1 \text{ loops}} + t + \underbrace{(t + t) + \cdots + (t + t)}_{\ell \text{ loops}} + \underbrace{(t + t) + \cdots + (t + t)}_{k_2 \text{ loops}} + t + t$$

Let both k_1 and $k_2 \to \infty$.

$$\phi\phi + \phi\phi = \frac{1}{\cosh(w)} \frac{\sinh(w)}{\sinh(2w)} + \frac{\sinh(w)}{\sinh(2w)} \frac{\sinh(w)}{\sinh(2w)} = \frac{3}{4\cosh^2 w}$$

Two-loops

$$I := \phi_{a\to b|\not{A}}\phi_{b\to a|\not{c}}, \quad II := \phi_{b\to c|\not{a}}\phi_{c\to b|\not{B}}$$

- k loops of I followed by l loops of II, with $k, l = 0, 1, \ldots,$ which gives

$$\sum_{k,l} I^k II^l = \frac{1}{1-I}\cdot\frac{1}{1-II};$$

- k_1 loops of I followed by l_1 loops of II, then followed by k_2 loops of I and finally followed by l_2 loops of II, with k_1, l_2 nonnegative and k_2, l_1 positive, which gives

$$\sum_{k_1, l_2=0, k_2, l_1=1}^{\infty} I^{k_1} II^{l_1} I^{k_2} II^{l_2} = \frac{I\cdot II}{(1-I)^2(1-II)^2};$$

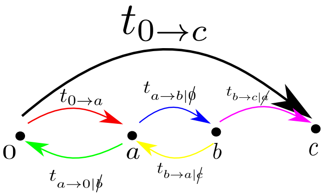
- the general term will be k_1 loops of $I \to l_1$ loops of $II \to \cdots \to k_n$ loops of $I \to l_n$ loops of II, with k_1, l_n nonnegative and the rest indices positive, which gives

$$\frac{(I\cdot II)^{n-1}}{(1-I)^n(1-II)^n}.$$

Therefore, loops I and II contribute as

$$\sum_{n=1}^{\infty}\frac{(I\cdot II)^{n-1}}{(1-I)^n(1-II)^n} = \frac{1}{1-(I+II)} = \sum_{k=0}^{\infty}(I+II)^k.$$

Two Loops



$$\phi = \frac{\phi \cdot \phi \cdot \phi}{1 - \phi\phi - \phi\phi}$$

Proposition. [LJ. and C. Vignat] For any positive integer n,

$$E_n\left(\frac{x}{6}\right) = \sum_{k=0}^{\infty} \frac{3^{k-n}}{4^{k+1}} E_n^{(2k+3)}\left(\frac{x}{2}+k\right).$$

In general

$$(x + 2c\mathcal{E} + c)^n = \sum_{k=0}^{\infty}\sum_{\ell=0}^{k} q_{k,\ell}\Big[x + 2(b-a)\mathcal{B} + 2(c-b)\mathcal{B}' + a\mathcal{E}^{(\ell)} + 2(b-a)\mathcal{U}^{(l)}$$
$$+ 2a\mathcal{U}'^{(k-\ell)} + 2(b-a)\mathcal{B}'^{(k-\ell)} + q'_{k,\ell}\Big]^n,$$

$$q_{k,\ell} := \binom{k}{\ell}\frac{(b-a)^{\ell+1}a^{k-\ell+1}(c-b)^{k-\ell}}{b^{k+1}(c-a)^{k-\ell+1}} \qquad q'_{k,\ell} = c + (2k-2\ell)b + (3\ell-k+1)a,$$

where

$$\left(\mathcal{E}^{(p)}+x\right)^n = E_n^{(p)}(x), \quad \left(\mathcal{B}^{(p)}+x\right)^n = B_n^{(p)}(x), \quad \mathcal{U}^n = \frac{1}{n+1}, \quad \mathcal{U}^{(p)} = \mathcal{U}_1 + \cdots + \mathcal{U}_p.$$

n loops?

Consider consecutive loops l_1, l_2, …, l_n, it seems like the contribution is

$$\sum_{k=0}^{\infty}\left(\sum_{\ell=1}^{n} l_\ell\right)^k = \frac{1}{1-(l_1+\cdots+l_n)}. \qquad (*)$$

- It feels right.
- I can "prove" it by induction.
- In general sites $0, 1, \ldots, N$:

$$\frac{1}{\cosh(Nw)} \overset{??}{=} \frac{\frac{1}{\cosh w}\cdot\left(\frac{\sinh w}{\sinh(2w)}\right)^N}{1-\left(\frac{1}{\cosh(w)}\frac{\sinh(w)}{\sinh(2w)}+(N-1)\frac{\sinh(w)}{\sinh(2w)}\frac{\sinh(w)}{\sinh(2w)}\right)}$$

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$$= \frac{\frac{1}{2^N\cosh^{N+1} w}}{1-\frac{N+3}{4}\cosh^N w}.$$

This shows $(*)$ is not correct.

$$\frac{1}{\cosh(Nw)} = \frac{1}{T_N(\cosh w)}.$$

Generalization

- Bessel process in $\mathbb{R}^n$:

$$R_t^{(n)} := \sqrt{\left(\tilde{W}_t^{(1)}\right)^2 + \cdots + \left(\tilde{W}_t^{(n)}\right)^2}$$

- Moment generating functions for hitting times:
  $H_z := \min_s \left\{ R_s^{(n)} = z \right\}$.

$$\mathbb{E}_x\left(e^{-\alpha H_z};\ \sup_{0\le s\le H_z} R_s^{(n)} < y\right) = \begin{cases} \frac{x^{-\nu} I_\nu(xw)}{z^{-\nu} I_\nu(zw)}, & 0 \le x \le z \le y; \\ \frac{S_\nu(yw,xw)}{S_\nu(yw,zw)}, & z \le x \le y, \end{cases}$$

- $n = 2 + 2\nu$ for $\nu \ge 0$

$$S_\nu\left(x,y\right) := \left(xy\right)^{-\nu}\left[I_\nu(x)K_\nu(y) - K_\nu(x)I_\nu(y)\right],$$

  and

$$I_\nu(x) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!\Gamma(\ell+\nu+1)}\left(\frac{x}{2}\right)^{2\ell+\nu}, \quad K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\nu\pi)}.$$

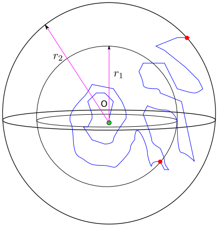
$n = 3 \Leftrightarrow \nu = 1/2$

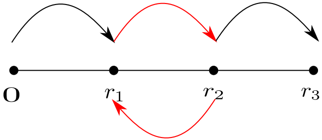
- $$I_{\frac{1}{2}}(x) = \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2m+\frac{1}{2}}}{m!\Gamma\left(m+\frac{3}{2}\right)} = \sqrt{\frac{2}{x\pi}}\sinh(x)$$

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t - 1}e^{tx} = \frac{te^{tx}e^{-\frac{t}{2}}}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} = \frac{te^{t\left(x-\frac{1}{2}\right)}}{2}\sinh\left(\frac{t}{2}\right)$$

$$K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}$$

$$\mathbb{E}_x\left(e^{-\alpha H_z}; \sup_{0\le s\le H_z} R_s^{(3)} < y\right) = \begin{cases} \frac{z\sinh(xw)}{x\sinh(zw)}, & 0 \le x \le z \le y \\ \frac{z\sinh((y-x)w)}{x\sinh((y-z)w)}, & z \le x \le y \end{cases}$$





$n = 3 \Leftrightarrow \nu = 1/2$

Let $r_1 = 1$, $r_2 = 2$, and $r_3 = 3$
Proposition. [LJ. and C. Vignat]

$$\frac{3^{n+1}}{n+1}\left[B_{n+1}\left(\frac{x}{6}+\frac{5}{6}\right) - B_{n+1}\left(\frac{x}{6}+\frac{1}{2}\right)\right] = \sum_{k\geq 0}\frac{3}{4}\left(\frac{1}{4}\right)^k E_n^{(2k+2)}\left(\frac{x+3+2k}{2}\right). \quad (\circledast)$$

Corollary 1. Take $x = 0$, $n = 2m-1$ in $(\circledast)$. The LHS is

$$\begin{aligned}\frac{3^{2m}}{2m}\left[B_{2m}\left(\frac{5}{6}\right) - B_{2m}\left(\frac{1}{2}\right)\right] &= \frac{3^{2m}}{2m}\left[\frac{1}{2}\left(1-2^{1-2m}\right)\left(1-3^{1-2m}\right)B_{2m} + \left(1-2^{1-2m}\right)B_{2m}\right]\\ &= \frac{3^{2m}}{2m}\left(1-2^{1-2m}\right)B_{2m}\left(\frac{1-3^{1-2m}}{2}+1\right)\\ &= \frac{3}{4m}\left(1-2^{1-2m}\right)\left(3^{2m}-1\right)B_{2m};\end{aligned}$$

while the RHS is

$$\sum_{k\geq 0}\frac{3}{4}\left(\frac{1}{4}\right)^k E_{2m-1}^{(2k+2)}\left(k+\frac{3}{2}\right).$$

Thus,

$$B_{2m} = \frac{m}{\left(1-2^{1-2m}\right)\left(3^{2m}-1\right)}\sum_{k\geq 0}\left(\frac{1}{4}\right)^k E_{2m-1}^{(2k+2)}\left(k+\frac{3}{2}\right).$$

$n = 3 \Leftrightarrow \nu = 1/2$

Proposition. [LJ. and C. Vignat]

$$\frac{3^{n+1}}{n+1}\left[B_{n+1}\left(\frac{x}{6}+\frac{5}{6}\right)-B_{n+1}\left(\frac{x}{6}+\frac{1}{2}\right)\right]=\sum_{k\geq 0}\frac{3}{4}\left(\frac{1}{4}\right)^k E_n^{(2k+2)}\left(\frac{x+3+2k}{2}\right). \qquad (\circledast)$$

Corollary 1.

$$B_{2m}=\frac{m}{\left(1-2^{1-2m}\right)\left(3^{2m}-1\right)}\sum_{k\geq 0}\left(\frac{1}{4}\right)^k E_{2m-1}^{(2k+2)}\left(k+\frac{3}{2}\right).$$

Corollary 2. Take $n=1$ in $(\circledast)$.

$$B_2(x)=x^2-x+\frac{1}{6}\Rightarrow \text{LHS}=\frac{x+1}{2},$$

$$\sum_{n=0}^{\infty}E_n^{(p)}(x)\frac{z^n}{n!}=\left(\frac{2}{e^z+1}\right)^p e^{xz}\Rightarrow E_1^{(2k+2)}(x)=x-(k+1).$$

$$\sum_{k\geq 0}\frac{3}{4}\left(\frac{1}{4}\right)^k\left(\frac{x+3+2k}{2}-k-1\right)=\sum_{k\geq 0}\frac{3}{4}\left(\frac{1}{4}\right)^k\left(\frac{x+1}{2}\right)=\frac{x+1}{2}.$$

Proposition. [LJ. and C. Vignat] (Two Loops) For any positive integer n,

$$3^n B_n\left(\frac{x+4}{6}\right) = \sum_{k=0}^{\infty} \frac{1}{2^k} E_n^{(2k+2)}\left(\frac{x+2k+3}{2}\right).$$

Several remarks are in order at this point:

- the identities obtained from this approach are not of the usual, convolutional type. Rather, they are connection-type identities between the usual Bernoulli and Euler polynomials and their higher-order counterparts;
- these inherently involve a mixture of higher-order Bernoulli and Euler polynomials;
- the interest of this approach is that each term in such a decomposition can be related to a physical object, namely one loop in a trajectory of a random process;
- this work should be considered as only a first approach to a more general project in which the richness of the possible setups for random walks is expected to generate a number of non-trivial identities about more general special functions.

End

Thank you!

Connection Coefficients for Higher-order Bernoulli and Euler Polynomials:

A Random Walk Approach

https://arxiv.org/abs/1809.04636