On Harmonic Sums: Integral and Matrix Representations with Connections to Partition-theoretic Generalization of Riemann Zeta-function and Random Walks

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Dalhousie University

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ANALYTIC AND COMBINATORIAL NUMBER THEORY: THE LEGACY OF RAMANUJAN: A CONFERENCE IN HONOR OF BRUCE C. BERNDT'S 80TH BIRTHDAY

Beginning-Partition

- Recall for the partition function $p(n)$, the generating function

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)}$$

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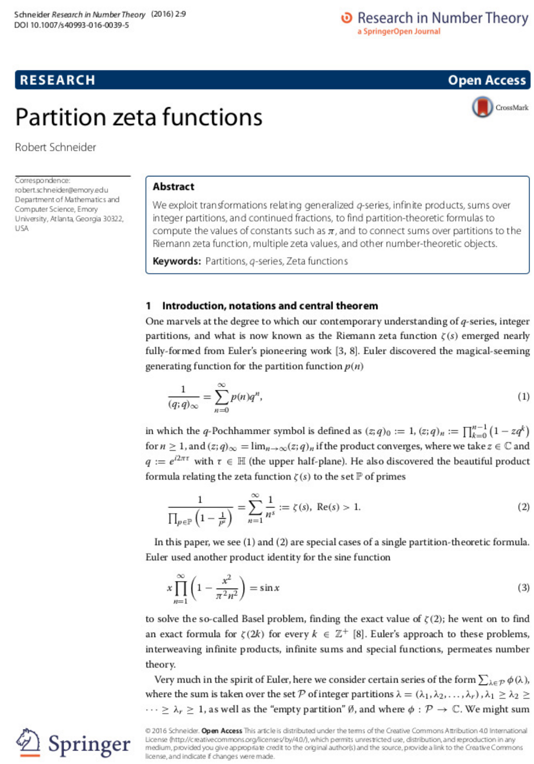

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Partition zeta functions

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Abstract

We exploit transformations relating generalized q-series, infinite products, sums over integer partitions, and continued fractions, to find partition-theoretic formulas to compute the values of constants such as π, and to connect sums over partitions to the Riemann zeta function, multiple zeta values, and other number-theoretic objects.

Keywords: Partitions, q-series, Zeta functions

1 Introduction, notations and central theorem

One marvels at the degree to which our contemporary understanding of q-series, integer partitions, and what is now known as the Riemann zeta function $\zeta(s)$ emerged nearly fully-formed from Euler's pioneering work [3, 8]. Euler discovered the magical-seeming generating function for the partition function $p(n)$

$$\frac{1}{(q;q)_\infty} = \sum_{n=0}^{\infty} p(n)q^n, \tag{1}$$

in which the q-Pochhammer symbol is defined as $(z;q)_0 := 1$, $(z;q)_n := \prod_{k=0}^{n-1}(1-zq^k)$ for $n \geq 1$, and $(z;q)_\infty = \lim_{n\to\infty}(z;q)_n$ if the product converges, where we take $z \in \mathbb{C}$ and $q := e^{i2\pi\tau}$ with $\tau \in \mathbb{H}$ (the upper half-plane). He also discovered the beautiful product formula relating the zeta function $\zeta(s)$ to the set $\mathbb{P}$ of primes

$$\frac{1}{\prod_{p\in\mathbb{P}}\left(1-\frac{1}{p^s}\right)} = \sum_{n=1}^{\infty}\frac{1}{n^s} := \zeta(s), \ \mathrm{Re}(s) > 1. \tag{2}$$

In this paper, we see (1) and (2) are special cases of a single partition-theoretic formula. Euler used another product identity for the sine function

$$x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{\pi^2 n^2}\right) = \sin x \tag{3}$$

to solve the so-called Basel problem, finding the exact value of $\zeta(2)$; he went on to find an exact formula for $\zeta(2k)$ for every $k \in \mathbb{Z}^+$ [8]. Euler's approach to these problems, interweaving infinite products, infinite sums and special functions, permeates number theory.

Very much in the spirit of Euler, here we consider certain series of the form $\sum_{\lambda\in\mathcal{P}}\phi(\lambda)$, where the sum is taken over the set $\mathcal{P}$ of integer partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$, as well as the "empty partition" $\emptyset$, and where $\phi : \mathcal{P} \to \mathbb{C}$. We might sum

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Partition zeta functions

Definition

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Notation: $a \mapsto s$:

$$\zeta_{\mathcal{P}}\left(\{a\}^k\right) := \sum_{\ell(\lambda)=k} \frac{1}{n_\lambda^a} = \sum_{\lambda_1 \geq \cdots \geq \lambda_k} \frac{1}{\lambda_1^a \cdots \lambda_k^a}$$

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Integral representation

Integral representation

Theorem (L. Jiu)

Let $m, k \in \mathbb{Z}_{>0}$ *and* $\xi_m = e^{2\pi i/m}$

$$S_{\boldsymbol{m}_k}(\infty) = \frac{(-1)^{mk}}{(m-1)!\,(mk)!}\int_0^1 \cdots \int_0^{1-x_1-\cdots-x_{k-2}} \log^{mk}\left(x_1^{\xi_m^0} x_2^{\xi_m^1} \cdots x_{m-1}^{\xi_m^{m-2}} (1-x_1-\cdots-x_{m-1})^{\xi_m^{m-1}}\right) dx_{m-1}\cdots dx_1$$

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Use the integral representation of multiple beta function $B(a_1,\ldots,a_m)$.

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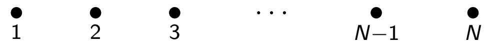

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$$\mathbb{P}(n_{k+1} \neq 1) \leq \left(1 - \frac{1}{N}\right)^k \longrightarrow 0 \quad as \quad k \to \infty$$

Transition matrix

$$\mathbf{S} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N} & \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \end{pmatrix}$$

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For $m, N \in \mathbb{Z}_{>0}$, define

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Theorem (L. Jiu and D. Shi)

Define three $N \times N$ matrices:

$$\mathbf{S}_f := \begin{pmatrix} f(1) & 0 & 0 & \cdots & 0 \\ f(2) & f(2) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f(N) & f(N) & f(N) & \cdots & f(N) \end{pmatrix}, \quad \mathbf{A}_f := \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ f(1) & 0 & \cdots & 0 & 0 \\ f(2) & f(2) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f(N-1) & f(N-1) & \cdots & f(N-1) & 0 \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

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$\mathbf{A}_f = \mathbf{\Delta} \mathbf{S}_f$, where $\mathbf{\Delta} := \left(\delta_{i-1,j}\right)_{N \times N}$

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Truncated version of identities

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Let

$$S\left(i_1, i_2, \ldots, i_k\right) = \sum_{n_1 \geq n_2 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1^{i} n_2^{i_2} \cdots n_k^{i_k}}$$

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Remark

$$S\left(i_1, i_2\right) = A\left(i_1, i_2\right) + A\left(i_1 + i_2\right)$$

and

$$S\left(i_1, i_2, i_3\right) = A\left(i_1, i_2, i_3\right) + A\left(i_1 + i_2, i_3\right) + A\left(i_1, i_2 + i_3\right) + A\left(i_1 + i_2 + i_3\right)$$

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Let

$$S\left(i_1, i_2, \ldots, i_k\right) = \sum_{n_1 \geq n_2 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1^{i} n_2^{i_2} \cdots n_k^{i_k}} \left(= S_{i_1,\ldots,i_k}(\infty)\right),$$

$$A\left(i_1, i_2, \ldots, i_k\right) = \sum_{n_1 > n_2 > \cdots n_k \geq 1} \frac{1}{n_1^{i} n_2^{i_2} \cdots n_k^{i_k}} \left(= H_{i_1,\ldots,i_k}(\infty)\right).$$

Then

$$\sum_{\sigma \in \Sigma_k} S\left(i_{\sigma(1)}, \ldots, i_{\sigma(k)}\right) = \sum_{\textit{partitions } \Pi \textit{ of } \{1,\ldots,k\}} c(\Pi)\zeta(\mathbf{i}, \Pi)$$

$$\sum_{\sigma \in \Sigma_k} A\left(i_{\sigma(1)}, \ldots, i_{\sigma(k)}\right) = \sum_{\textit{partitions } \Pi \textit{ of } \{1,\ldots,k\}} \tilde{c}(\Pi)\zeta(\mathbf{i}, \Pi)$$

Remark

$$S\left(i_1, i_2\right) = A\left(i_1, i_2\right) + A\left(i_1 + i_2\right)$$

and

$$S\left(i_1, i_2, i_3\right) = A\left(i_1, i_2, i_3\right) + A\left(i_1 + i_2, i_3\right) + A\left(i_1, i_2 + i_3\right) + A\left(i_1 + i_2 + i_3\right)$$

Theorem (L. Jiu and D. Shi)

$$S(f, g; N-1, m) = A(f, g; N, m) + A(fg; N, m)$$

$$S(f, g, h; N-1, m) = A(f, g, h; N, m) + A(fg, h; N, m) + A(f, gh; N, m) + A(fgh; N, m)$$

Truncated version of identities

Theorem (M. Hoffman)

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by only using $\mathbf{A}_f = \mathbf{\Delta S}_f$

Diagonalization

$$\mathbf{S}_f := \begin{pmatrix} f(1) & 0 & 0 & \cdots & 0 \\ f(2) & f(2) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f(N) & f(N) & f(N) & \cdots & f(N) \end{pmatrix}$$

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Theorem (L. Jiu and D. Shi)

$\boldsymbol{S}_f$ has eigenvalues $\{f(1), \ldots, f(N)\}$. If all of them are distinct, define $\boldsymbol{D}_f = (d_{i,j})_{N\times N}$ and $\boldsymbol{E}_f = (e_{i,j})_{N\times N}$ by

- *if $i \geq j$,*

$$d_{i,j} = \frac{f(i)}{f(N)} \prod_{k=i+1}^{N} \left(1 - \frac{f(k)}{f(j)}\right) \qquad \text{and} \qquad e_{i,j} = \frac{f(N)}{f(i)} \prod_{\substack{k=j \\ k\neq i}}^{N} \frac{1}{1 - \frac{f(k)}{f(i)}};$$

- *if $i < j$, $d_{i,j} = 0 = e_{i,j}$*

Then, $\boldsymbol{D}_f^{-1} = \boldsymbol{E}_f$ and

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Example

For $\boldsymbol{S}_{1_k}(N)$,

$$\mathbf{S}_f := \begin{pmatrix} f(1) & 0 & 0 & \cdots & 0 \\ f(2) & f(2) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f(N) & f(N) & f(N) & \cdots & f(N) \end{pmatrix}$$

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For $\boldsymbol{S}_{1_k}(N)$, let $f(x) = 1/x$, then

$$\boldsymbol{S}_{1_k}(N) = N\left(\mathbf{S}_f^{k+1}\right)_{N,1}$$

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