Orthogonal Polynomials for Higher-Order Euler Polynomials

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Main object

Definition. The Euler polynomial of order p, denoted by $E_n^{(p)}(x)$, is defined by

$$\left(\frac{2}{e^t+1}\right)^p e^{xt} = \sum_{n=0}^{\infty} E_n^{(p)}(x)\frac{t^n}{n!}.$$

- When $p=1$, $E_n^{(1)}(x) = E_n(x)$ are the (usual) Euler polynomials

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}.$$

- $E_n = 2^n E_n(1/2)$ are the Euler numbers

$$\frac{2}{e^t+e^{-t}} = \sum_{n=0}^{\infty} E_n\frac{t^n}{n!}.$$

Table

$$\left(\frac{2}{e^z+1}\right)^p e^{xz} = \sum_{n=0}^{\infty} E_n^{(p)}(x)\frac{z^n}{n!}.$$

$$E_n^{(1)}(x) = E_n(x) \qquad E_n = 2^n E_n(1/2)$$

	$p=1$	$p=2$	$p=3$
$n=0$	1	1	1
$n=1$	$x-\frac{1}{2}$	$x-1$	$x-\frac{3}{2}$
$n=2$	x^2-x	$x^2-2x+\frac{1}{2}$	$x^2-3x+\frac{3}{2}$
$n=3$	$x^3-\frac{3}{2}x^2+\frac{1}{4}$	$x^3-3x^2+\frac{3}{2}x+\frac{1}{2}$	$x^3-\frac{9}{2}x^2+\frac{9}{2}x$
$n=4$	x^4-2x^3+x	$x^4-4x^3+3x^2+2x-1$	$x^4-6x^3+9x^2-3$

Random variable

Let X be a random variable with density function $p(t)$ on $\mathbb{R}$ and with moments m_n, i.e.,

$$m_n = \mathbb{E}[X^n] = \int_{\mathbb{R}} t^n p(t)\mathrm{d}t.$$

Let $P_n(y)$ be the monic orthogonal polynomials with respect to X (or w. r. t. m_n), i.e., $\deg P_n = n$, $\mathrm{LC}[P_n] = 1$, and

$$\int_{\mathbb{R}} P_m(t)P_n(t)p(t)\mathrm{d}t = c_n\delta_{m,n} = \begin{cases} c_n, & \text{if } m = n; \\ 0, & \text{otherwise.} \end{cases}$$

Equivalently, for all $0 \leq r < n$

$$\boxed{y^r P_n(y)\Big|_{y^k = m_k} = 0.}$$

P_n satisfies a three-term recurrence: for some sequences $(s_n)_{n\geq 0}$ and $(t_n)_{n\geq 1}$,

$$P_{n+1}(y) = (y - s_n)P_n(y) - t_nP_{n-1}(y).$$

Hankel determinants

$$y^r P_n(y)\Big|_{y^k=m_k} = 0 \quad P_{n+1}(y) = (y - s_n)P_n(y) - t_n P_{n-1}(y).$$

$$P_n(y) = \frac{\det\begin{pmatrix} m_0 & m_1 & \cdots & m_n \\ m_1 & m_2 & \cdots & m_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n-1} & m_n & \cdots & m_{2n-1} \\ 1 & y & \cdots & y^n \end{pmatrix}}{\det\begin{pmatrix} m_0 & m_1 & \cdots & m_{n-1} \\ m_1 & m_2 & \cdots & m_n \\ \vdots & \vdots & \ddots & \vdots \\ m_{n-1} & m_n & \cdots & m_{2n-2} \end{pmatrix}}.$$

Orthogonal polynomials

Carlitz [3, eq. 4.7] and also with Al-Salam [1, p. 93] gave the monic orthogonal polynomials, denoted by $Q_n(y)$, with respect to E_n. More precisely, they obtained $Q_0(y) = 1$, $Q_1(y) = y$ and for $n \geq 1$,

$$Q_{n+1}(y) = yQ_n(y) + n^2 Q_{n-1}(y).$$

Example

$$\frac{2}{e^z + e^{-z}} = \sum_{n=0}^{\infty} E_n \frac{z^n}{n!}$$

n	0	1	2	3	4
E_n	1	0	-1	0	5

$$Q_2(y) = yQ_1(y) + 1^2 Q_0(y) = y^2 + 1$$

- $$y^0 Q_2(y)\Big|_{y^k=E_k} = (y^2+1)\Big|_{y^k=E_k} = E_2 + 1 = 0;$$
- $$yQ_2(y)\Big|_{y^k=E_k} = (y^3+y)\Big|_{y^k=E_k} = 0.$$

Probabilistic interpretation

Let L_E be a random variable with density function $p_E(t) := \operatorname{sech}(\pi t)$ on $\mathbb{R}$.

- $$\mathbb{E}\left[L_E^n\right] = \int_{\mathbb{R}} t^n \operatorname{sech}(\pi t) \mathrm{d}t = \frac{(-1)^{\frac{n}{2}} E_n}{2^n} \Longrightarrow \mathbb{E}\left[(2iL_E)^n\right] = E_n$$
- Also consider a sequence of independent and identically distributed (i. i. d.) random variables $(L_{E_i})_{i=1}^p$ with each L_{E_i} having the same distribution as L_E. Then $E_n^{(p)}(x)$ is the nth moment of a certain random variable:

$$E_n^{(p)}(x) = \mathbb{E}\left[\left(x + \sum_{i=1}^{p} iL_{E_i} - \frac{p}{2}\right)^n\right].$$

Main result

E_n	$Q_{n+1}(y) = yQ_n(y) + n^2 Q_{n-1}(y)$
$E_n(x)$	???
$E_n^{(p)}(x)$	???

Let $\Omega_n^{(p)}(y)$ be the orthogonal polynomials with respect to $E_n^{(p)}(x)$, i.e., for any $0 \leq r < n$,

$$y^r \Omega_n^{(p)}(y)\Big|_{y^k = E_k^{(p)}(x)} = 0.$$

Also let $\Omega_n(y) = \Omega_n^{(1)}(y)$ be the orthogonal polynomials w. r. t. $E_n(x)$.

Theorem (L. Jiu and D. Shi)

$$\Omega_{n+1}^{(p)}(y) = \left(y - x + \frac{p}{2}\right)\Omega_n^{(p)}(y) + \frac{n(n+p-1)}{4}\Omega_{n-1}^{(p)}(y).$$

$$\Omega_n^{(p)}(y) = \frac{i^n n!}{2^n} P_n^{\left(\frac{p}{2}\right)}\left(-i\left(y - x + \frac{p}{2}\right); \frac{\pi}{2}\right).$$

The *Meixner-Pollaczek polynomials* are defined by

$$P_n^{(\lambda)}(y;\phi) := \frac{(2\lambda)_n}{n!} e^{in\phi}\, {}_2F_1\left(\left.\begin{matrix} -n, \lambda+iy \\ 2\lambda \end{matrix}\right| 1-e^{-2i\phi}\right),$$

where

$$(x)_n := x(x+1)(x+2)\cdots(x+n-1),$$

and

$${}_2F_1\left(\left.\begin{matrix} a, b \\ c \end{matrix}\right| t\right) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n} \cdot \frac{t^n}{n!}.$$

E_n	$Q_{n+1}(y) = yQ_n(y) + n^2 Q_{n-1}(y)$
$E_n(x)$	$\Omega_{n+1}(y) = \left(y - x + \frac{1}{2}\right)\Omega_n^{(p)}(y) + \frac{n^2}{4}\Omega_{n-1}^{(p)}(y).$
$E_n^{(p)}(x)$	$\Omega_{n+1}^{(p)}(y) = \left(y - x + \frac{p}{2}\right)\Omega_n^{(p)}(y) + \frac{n(n+p-1)}{4}\Omega_{n-1}^{(p)}(y).$

$$Q_n \xrightarrow{K1} \Omega_n \xrightarrow{K2} \Omega_n^{(p)}$$

Key 1 (K1)

Lemma (L. Jiu and D. Shi)

random variable	*moments*	*monic orthogonal polynomial*
X	m_n	$P_n(y): \quad P_{n+1}(y) = (y - s_n)P_n(y) - t_n P_{n-1}(y)$
$X + c$	$\sum_{k=0}^{n} \binom{n}{k} m_k c^{n-k}$	$\bar{P}_n(y): \quad \bar{P}_{n+1}(y) = (y - s_n - c)\bar{P}_n(y) - t_n \bar{P}_{n-1}(y)$
CX	$C^n m_n$	$\tilde{P}_n(y): \quad \tilde{P}_{n+1}(y) = (y - Cs_n)\tilde{P}_n(y) - C^2 t_n \tilde{P}_{n-1}(y)$

Recall that

$$\mathbb{E}\left[(2iL_E)^n\right] = E_n \quad \text{and} \quad E_n(x) = \mathbb{E}\left[\left(x + iL_E - \frac{1}{2}\right)^n\right].$$

$$X = iL_E, \quad C = \frac{1}{2}, \quad c = x - \frac{1}{2}.$$

Key 2 (K2)

For the *Meixner-Pollaczek polynomials*

$$P_n^{(\lambda)}(y;\phi) := \frac{(2\lambda)_n}{n!} e^{in\phi} \, {}_2F_1\left(\left. \begin{matrix} -n, \lambda + iy \\ 2\lambda \end{matrix} \right| 1 - e^{-2i\phi} \right),$$

KEY.

$$P_n^{(\lambda+\mu)}(y_1 + y_2, \phi) = \sum_{k=0}^{n} P_k^{(\lambda)}(y_1, \phi) \, P_{n-k}^{(\mu)}(y_2, \phi).$$

$$Q_n(y) = i^n n! P_n^{\left(\frac{1}{2}\right)}\left(\frac{-iy}{2}; \frac{\pi}{2}\right)$$

$$\Longrightarrow \Omega_n^{(p)}(y) = \frac{i^n n!}{2^n} P_n^{\left(\frac{p}{2}\right)}\left(-i\left(y - x + \frac{p}{2}\right); \frac{\pi}{2}\right)$$

Bernoulli Polynomials

Bernoulli polynomials $B_n(x)$ and Bernoulli numbers $B_n = B_n(0)$:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

Theorem (L. Jiu and D. Shi)

Let $\varrho_n(y)$ be the orthogonal polynomials with respect to $B_n(x)$, Then,

$$\varrho_{n+1}(y) = \left(y - x + \frac{1}{2}\right) \varrho_n(y) + \frac{n^4}{4(2n+1)(2n-1)} \varrho_{n-1}(y).$$

In particular,

$$\varrho_n(y) = \frac{n!}{(n+1)_n} p_n\left(y; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right),$$

where $p_n(y; a, b, c, d)$ is the continuous Hahn polynomial.

Bernoulli Polynomials

Generalization to $B_n^{(p)}(x)$:

$$\left(\frac{t}{e^t-1}\right)^p e^{tx} = \sum_{n=0}^{\infty} B_n^{(p)}(x)\frac{t^n}{n!}?$$

The key property for Meixner-Pollaczek polynomials

$$P_n^{(\lambda+\mu)}(y_1+y_2,\phi) = \sum_{k=0}^{n} P_k^{(\lambda)}(y_1,\phi)\,P_{n-k}^{(\mu)}(y_2,\phi)$$

does not hold for continuous Hahn polynomials.

random variable	moments	monic orthogonal polynomial
X	m_n	$P_n(y): \quad P_{n+1}(y) = (y-s_n)P_n(y) - t_nP_{n-1}(y)$
$X+c$	$\sum_{k=0}^{n}\binom{n}{k}m_kc^{n-k}$	$\bar{P}_n(y): \quad \bar{P}_{n+1}(y) = (y-s_n-c)\bar{P}_n(y) - t_n\bar{P}_{n-1}(y)$
CX	C^nm_n	$\tilde{P}_n(y): \quad \tilde{P}_{n+1}(y) = (y-Cs_n)\tilde{P}_n(y) - C^2t_n\tilde{P}_{n-1}(y)$
$X+Y$	Convolution	???

Conjecture on $B_n^{(p)}(x)$

Let $\varrho_{n+1}^{(p)}(y)$ be the monic orthogonal polynomial with respect to $B_n^{(p)}(x)$, and assume the three-term recurrence is

$$\varrho_{n+1}^{(p)}(y) = \left(y - a_n^{(p)}\right) \varrho_n^{(p)}(y) + b_n^{(p)} \varrho_{n-1}(y).$$

Proposition. [L. Jiu and D. Shi] $a_n^{(p)} = x - p/2$.

The first several terms of $b_n^{(p)}$ is given in the following table

	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$
$n=1$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{5}{12}$
$n=2$	$\frac{4}{15}$	$\frac{13}{30}$	$\frac{3}{5}$	$\frac{23}{30}$	$\frac{14}{15}$
$n=3$	$\frac{81}{140}$	$\frac{372}{455}$	$\frac{1339}{1260}$	$\frac{2109}{1610}$	$\frac{1527}{980}$
$n=4$	$\frac{64}{63}$	$\frac{3736}{2821}$	$\frac{138688}{84357}$	$\frac{668543}{339549}$	$\frac{171830}{74823}$
$n=5$	$\frac{625}{396}$	$\frac{1245075}{636988}$	$\frac{299594775}{127670972}$	$\frac{42601023200}{15509529057}$	$\frac{3638564965}{1154491404}$

The first column has formula $\frac{n^4}{4(2n+1)(2n-1)}$

Conjecture on $B_n^{(p)}(x)$

	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$
$n=1$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{5}{12}$
$n=2$	$\frac{4}{15}$	$\frac{13}{30}$	$\frac{3}{5}$	$\frac{23}{30}$	$\frac{14}{15}$
$n=3$	$\frac{81}{140}$	$\frac{372}{455}$	$\frac{1339}{1260}$	$\frac{2109}{1610}$	$\frac{1527}{980}$
$n=4$	$\frac{64}{63}$	$\frac{3736}{2821}$	$\frac{138688}{84357}$	$\frac{668543}{339549}$	$\frac{171830}{74823}$
$n=5$	$\frac{625}{396}$	$\frac{1245075}{636988}$	$\frac{299594775}{127670972}$	$\frac{42601023200}{15509529057}$	$\frac{3638564965}{1154491404}$

Conjecture (K. Dilcher)

$$b_3^{(p)} = \frac{175p^2 + 315p + 158}{140(2p+3)};$$

$$b_4^{(p)} = \frac{6125p^4 + 25725p^3 + 41965p^2 + 29547p + 7230}{21(5p+3)(175p^2 + 315p + 158)};$$

$$b_5^{(p)} = 25(5p+3)(471625p^6 + 3678675p^5 + 12324235p^4 + 22096305p^3 + 22009540p^2 + 11549748p + 2519472) \Big/ (132(175p^2 + 315p + 158)(6125p^4 + 25725p^3 + 41965p^2 + 29547p + 7230)).$$

Thank you!

W. A. Al-Salam and L. Carlitz, Some determinants of Bernoulli, Euler and related numbers, *Port. Math*. **18** (1959), 91–99.

W. A. Al-Salam and T. S. Chihara, Convolutions of orthonormal polynomials, *SIAM J. Math. Anal*. **7** (1976), 16–28.

L. Carlitz, Bernoulli and Euler, numbers and orthogonal polynomials, *Duke Math. J*. **26** (1959), 1–16.

H. Chen, Bernoulli numbers via determinants, *Internat. J. Math. Ed. Sci. Tech*. **34** (2003), 291–297.

A. Dixit, V. H. Moll, and C. Vignat, The Zagier modification of Bernoulli numbers and a polynomial extension. Part I, *Ramanujan J*. **33** (2014), 379–422.

P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge Univ. Press, 2010.

A. Hamdi and J. Zeng, Orthogonal polynomials and operator orderings, *J. Math. Phys*. 51 (2010), Article 043506.

- M. E. H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable*, Cambridge Univ. Press, 2005.
- L. Jiu, V. H. Moll, and C. Vignat, Identities for generalized Euler polynomials, *Integral Transforms Spec. Funct.* **25** (2014), 777–789.
- R. Koekoek, P. A. Lesky, and R. F. Swarttouw, *Hypergeometric Orthogonal Polynomials and Their q-Analogues*, Springer, 2010.
- C. Krattenthaler, Advanced determinant calculus, Séminaire Lotharingien *Combin.* **42** (1999), Article B42q.
- H. Liang, L. Mu, and Y. Wang, Catalan-like numbers and Stieltjes moment sequences, *Discrete Math.* **339** (2016), 484–488.
- F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, (eds.), *NIST Handbook of Mathematical Functions*, Cambridge Univ. Press, 2010.
- J. A. Shohat and J. D. Tamarkin, *The Problem of Moments*, AMS, 1943.
- R. P. Stanley, *Catalan numbers*, Cambridge Univ. Press, 2015.

J. Touchard, Nombres exponentiels et nombres de Bernoulli, *Canad. J. Math.* **8** (1956), 305–320.