# On $b$-ary Binomial Coefficients

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# Outlines

1. Digital Binomial Identity & $\binom{n}{k}_b$
   - Binary expansion
   - $b$-ary binomial coefficients $\binom{n}{k}_b$
   - Some properties
2. Negative Entries
   - Binomial coefficients with negative entries
   - Generalization of $b$-ary binomial coefficients
   - Results and CONJECTURES

# Binary expansion

## Example

$$10 = 8 + 2 = 2^3 + 2^1 = (1010)_2$$

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $(n)_2$ | $(000)_2$ | $(001)_2$ | $(010)_2$ | $(011)_2$ | $(100)_2$ | $(101)_2$ | $(110)_2$ |

$$(X+Y)^6 = \sum_{k=0}^{6} \binom{6}{k} X^k Y^{6-k}$$

## Definition

$S_2(n) = \#$ of 1's in the binary expansion of $n$.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $(n)_2$ | $(000)_2$ | $(001)_2$ | $(010)_2$ | $(011)_2$ | $(100)_2$ | $(101)_2$ | $(110)_2$ |
| $S_2(n)$ | 0 | 1 | 1 | 2 | 1 | 2 | 2 |

# Digital binomial identity

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $(n)_2$ | $(000)_2$ | $(001)_2$ | $(010)_2$ | $(011)_2$ | $(100)_2$ | $(101)_2$ | $(110)_2$ |
| $S_2(n)$ | 0 | 1 | 1 | 2 | 1 | 2 | 2 |

$$(X+Y)^{S_2(6)} = (X+Y)^2 = X^2Y^0 + XY + XY + X^0Y^2$$

$$(X+Y)^{S_2(6)} = \sum_{k=0,2,4,6} X^{S_2(k)}Y^{S_2(6-k)}$$

## Remark ("Carry")

| $k+(n-k)$ | $0+6$ | $1+5$ | $2+4$ | $3+3$ | $4+2$ | $5+1$ | $6+0$ |
|---|---|---|---|---|---|---|---|
| Binary | 000<br>110<br>110 | 001<br>101<br>110 | 010<br>100<br>110 | 011<br>011<br>110 | 100<br>010<br>110 | 101<br>001<br>110 | 110<br>000<br>110 |
| Carry-free | ✓ | x | ✓ | x | ✓ | x | ✓ |

## Theorem

$$(X+Y)^{S_2(n)} = \sum_{0\leq k \lesssim_2 n} X^{S_2(k)}Y^{S_2(n-k)},$$

$$0 \leq k \lesssim_2 n = \begin{cases} 0 \leq k \leq n \\ k+(n-k) \text{ carry free} \end{cases}$$

# $b$-ary

## Definition

$$S_b(n) = \text{sum of all digits of } n \text{ in its expansion of base } b.$$

## Example

$6 = (110)_2 \Rightarrow S_2(6) = 2$ $6 = (12)_4 \Rightarrow S_4(6) = 1 + 2 = 3.$

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $(n)_4$ | $(00)_4$ | $(01)_4$ | $(02)_4$ | $(03)_4$ | $(10)_4$ | $(11)_4$ | $(12)_4$ |
| $S_4(n)$ | 0 | 1 | 2 | 3 | 1 | 2 | 3 |

$$(X+Y)^{S_4(6)} = (X+Y)^3 = X^3 + 3X^2Y + 3XY^2 + Y^3$$

## Remark

Only summing over $0 \le k \lesssim_4 6$ is not enough.

# $b$-ary digital binomial identity

## Question

$$(X+Y)^{S_2(n)} = \sum_{0\leq k\lesssim_2 n} X^{S_2(k)} Y^{S_2(n-k)}$$

$$\Downarrow$$

$$(X+Y)^{S_b(n)} = \sum_{0\leq k\lesssim_b n} ???X^{S_b(k)} Y^{S_b(n-k)}$$

## Lemma (L. J, C. Vignat)

$$(X+Y)^{S_b(n)} = \sum_{0\leq k\lesssim_b n} \left( \prod_{j=0}^{b-1} (j!)^{S_b^{(j)}(n)-S_b^{(j)}(k)-S_b^{(j)}(n-k)} \right) X^{S_b(k)} Y^{S_b(n-k)},$$

*where*

$$S_b^{(j)}(m) = \#\text{ of } j\text{'s in } m\text{'s } b\text{-ary expansion.}$$

# $b$-ary binomial coefficients

## Definition

$$\binom{n}{k}_b := \begin{cases} \prod_{j=0}^{b-1} (j!)^{S_b^{(j)}(n) - S_b^{(j)}(k) - S_b^{(j)}(n-k)} & \text{if carry-free} \\ 0 & \text{otherwise} \end{cases}$$

## Remark

$$(X+Y)^{S_b(n)} = \sum_{k=0}^{n} \binom{n}{k}_b X^{S_b(k)} Y^{S_b(n-k)}.$$

## Theorem (L. J, C. Vignat)

$$\binom{n}{k}_b = \prod_{\ell=0}^{N-1} \binom{n_\ell}{k_\ell}, \quad \begin{cases} n = (n_{N-1} \cdots n_0)_b \\ k = (k_{N-1} \cdots k_0)_b \end{cases}$$

"carry-free" $\Leftrightarrow k_\ell \leq n_\ell$, $\ell = 0, \ldots, N-1$.

# Generating Function

## Definition

The generating function of the $b$-ary binomial coefficients is defined as

$$f(n,b,x) := \sum_{k=0}^{n} \binom{n}{k}_b x^k.$$

## Example ($b = 4$)

| $n$ | $f(n,4,x)$ | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | $1+x$ | 3 | $(1+x)^3$ | 5 | $(1+x)\left(1+x^4\right)$ | 7 | $(1+x)^3\left(1+x^4\right)$ |
| 2 | $(1+x)^2$ | 4 | $1+x^4$ | 6 | $(1+x)^2\left(1+x^4\right)$ | 8 | $\left(1+x^4\right)^2$ |

## Theorem (L. J, C. Vignat)

*Let* $n = (n_{N-1}\cdots n_0)_b$. *Then,*

$$f(n,b,x) = \sum_{k=0}^{n} \binom{n}{k}_b x^k = \prod_{\ell=0}^{N-1} \left(1+x^{b^\ell}\right)^{n_\ell}.$$

# Lucas' Theorem

## Theorem (Lucas)

*For a prime p,*

$$\binom{n}{k} \equiv \prod_{\ell=0}^{N-1} \binom{n_\ell}{k_\ell} = \binom{n}{k}_p \mod p$$

## Proof.

A simple proof is obtained by noting that

$$\sum_{k=0}^{n} \binom{n}{k}_p x^k = \prod_{\ell=0}^{N-1} \left(1 + x^{p^\ell}\right)^{n_\ell} \equiv (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k, \quad \mod p$$

when expanding

$$n = \sum_{l=0}^{N-1} n_\ell p^\ell.$$

□

## Example ($b = 4$)

1

1 1

1 2 1

1 3 3 1

1 . . . 1

1 1 . . 1 1

1 2 1 . 1 2 1

1 3 3 1 1 3 3 1

1 . . . 2 . . . 1

1 1 . . 2 2 . . 1 1

1 2 1 . 2 4 2 . 1 2 1

1 3 3 1 2 6 6 2 1 3 3 1

1 . . . 3 . . . 3 . . . 1

1 1 . . 3 3 . . 3 3 . . 1 1

1 2 1 . 3 6 3 . 3 6 3 . 1 2 1

1 3 3 1 3 9 9 3 3 9 9 3 1 3 3 1

## Fact

$$\binom{n}{k}_b = \prod_{\ell=0}^{N-1} \binom{n_\ell}{k_\ell} = \binom{n_{N-1}}{k_{N-1}} \prod_{l=0}^{N-2} \binom{n_\ell}{k_\ell}.$$

# Chu-Vandermonde identity

## Theorem (L. J, C. Vignat)

*Suppose that the addition of $m$ and $n$ is carry-free in base $b$, then*

$$\binom{m+n}{r}_b = \sum_{0 \le k \lesssim_b r} \binom{m}{k}_b \binom{n}{r-k}_b .$$

## Remark

Both $m+n$ is carry free and $0 \le k \lesssim_b n$ condition are necessary. Let $b = 2$

1. $m = n = r = 1 \Rightarrow m + n = 2$ is not carry-free.

$$LHS = \binom{2}{1}_2 = \binom{1}{0}\binom{0}{1} = 0 \neq \sum_{0 \le k \lesssim_2 1} \binom{1}{k}_2 \binom{1}{1-k}_2 = 2.$$

2. The case $m = r = 2$ and $n = 1$ also shows that $0 \le k \lesssim_b r$ cannot be replaced by $0 \le k \le r$.

# Other properties

- $$\binom{n}{k}_b = \binom{n}{n-k}_b$$
- When $\binom{n}{k}_b \neq 0$,
$$\binom{n}{k}_b = \binom{n-1}{k-1}_b + \binom{n-1}{k}_b$$
and
$$\binom{n}{k}_b = \binom{n-b^j}{k-b^j}_b + \binom{n-b^j}{k}_b$$
- $$\sum_{k=0}^{n} \binom{n}{k}_b \binom{k}{j}_b (-1)^{S_b(k)+S_b(j)} = \delta_{n,j} = \begin{cases} 1, & n=j; \\ 0 & \text{otherwise.} \end{cases}$$

# Binomial Coefficients

For $\alpha \in \mathbb{C}$,

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} \Longrightarrow (1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n.$$

**Example.**

$$\binom{-2}{4} = \frac{(-2)(-3)(-4)(-5)}{4!} = 5.$$

Question:

$$\binom{-2}{-4} = ?$$

Answer:

$$\binom{x}{y} := \lim_{\varepsilon \to 0} \frac{\Gamma(x+1+\varepsilon)}{\Gamma(y+1+\varepsilon)\,\Gamma(x-y+1+\varepsilon)} \qquad \boxed{n! = \Gamma(n+1)}.$$

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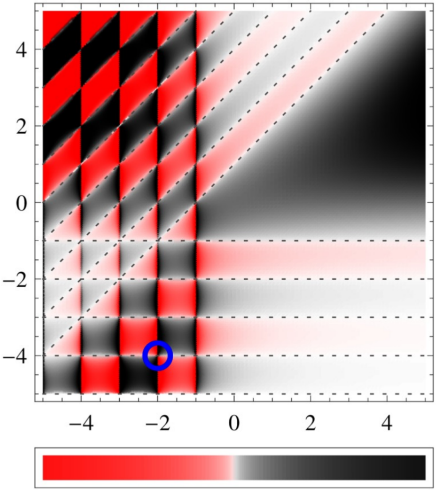

This scale is also visible along the line $y = 1$.

This is a plot of:

$$\binom{x}{y} := \frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)}$$

Defined and smooth on $\mathbb{R}\backslash\{x = -1, -2, \ldots\}$.

Directional limits exist at integer points:

$$\lim_{\varepsilon \to 0} \binom{-2+\varepsilon}{-4+r\varepsilon} = \frac{1}{2!} \lim_{\varepsilon \to 0} \frac{\Gamma(-1+\varepsilon)}{\Gamma(-3+r\varepsilon)} = 3r$$

since $\Gamma(-n+\varepsilon) = \frac{(-1)^n}{n!} \frac{1}{\varepsilon} + O(1)$

**DEF** For all $x, y \in \mathbb{Z}$:

$$\binom{x}{y} := \lim_{\varepsilon \to 0} \frac{\Gamma(x+1+\varepsilon)}{\Gamma(y+1+\varepsilon)\Gamma(x-y+1+\varepsilon)}$$

Daniel E. Loeb, Sets with a negative number of elements, Adv. Math. 91 (1992), 64–74.

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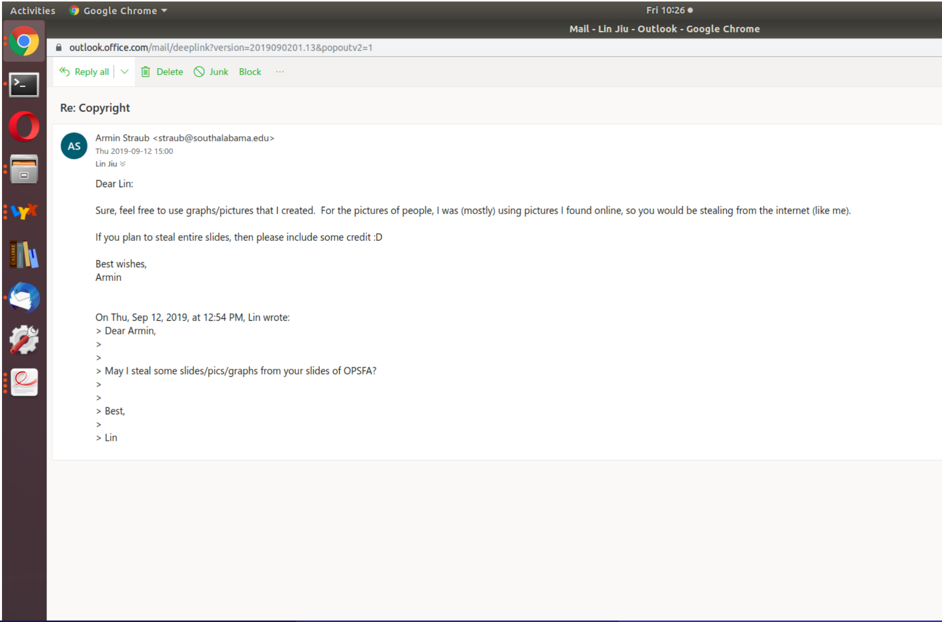

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**Re: Copyright**

AS
Armin Straub <straub@southalabama.edu>
Thu 2019-09-12 15:00
Lin Jiu

Dear Lin:

Sure, feel free to use graphs/pictures that I created. For the pictures of people, I was (mostly) using pictures I found online, so you would be stealing from the internet (like me).

If you plan to steal entire slides, then please include some credit :D

Best wishes,
Armin

On Thu, Sep 12, 2019, at 12:54 PM, Lin wrote:
> Dear Armin,
>
>
> May I steal some slides/pics/graphs from your slides of OPSFA?
>
>
> Best,
>
> Lin

# Negative entries

## Theorem (D. E. Loeb)

*For $n \in \mathbb{Z}$,*

$$\binom{n}{k} := \left[x^k\right](1+x)^n,$$

*where, when $k$ is a negative integer, it is the coefficient of $x^k$ of the inverse power series of $(1+x)^n$. Namely, letting $n \in \mathbb{N}$,*

$$(1+x)^{-n} = \frac{1}{(1+x)^n} = \sum_{k=0}^{\infty}\binom{-n}{k}x^k = \sum_{j=n}^{\infty}\binom{-n}{-j}\frac{1}{x^j}.$$

1. *If $n \geqslant k \geqslant 0$, then $\binom{n}{k} = \binom{n}{k}$.*
2. *If $k \geqslant 0 > n$, then $\binom{n}{k} = (-1)^k\binom{n+k-1}{k}$.*
3. *If $0 > n > k$, then $\binom{n}{k} = (-1)^{n+k}\binom{-k-1}{n-k}$.*
4. *If $k > n \geqslant 0$, or $0 > k > n$, or $k \geqslant 0 > n$, then $\binom{n}{k} = 0$.*

# Example

Let $n = 2$.

$$\frac{1}{1+x} = \frac{1}{x} \cdot \frac{1}{1+\frac{1}{x}} = \frac{1}{x}\sum_{j=0}^{\infty}\frac{(-1)^j}{x^j} = \sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{x^j}$$

$$\Rightarrow \frac{1}{(1+x)^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{x^j}\right) = \sum_{j=2}^{\infty}\frac{(-1)^{j-1}j}{x^{j+1}}$$

This shows that

$$\binom{-2}{-4} = (-1)^{3-1}\cdot 3 = 3.$$

$$\binom{-2}{-4} = (-1)^{-2-4}\binom{4-1}{-2-(-4)} = \binom{3}{2} = 3.$$

# $b$-ary

## Definition

Let $n, k \in \mathbb{Z}$ with $|n| = (n_{N-1} \cdots n_1 n_0)_b$ and $|k| = (k_{N-1} \cdots k_1 k_0)_b$.

- Define $S_b(n) = \operatorname{sign}(n) S_b(|n|)$.
- The three types of $b$-ary binomial coefficients are defined as

$$\binom{n}{k}_b^{(1)} := \prod_{\ell=0}^{N-1} \binom{\operatorname{sign}(n) n_\ell}{k_\ell},$$

$$\binom{n}{k}_b^{(2)} := \left[x^k\right] \prod_{\ell=0}^{N-1} \left(1 + x^{b^\ell}\right)^{\operatorname{sign}(n) n_\ell}$$

$$\binom{n}{k}_b^{(3)} := \left[X^{S_b(n) - S_b(k)} Y^{S_b(k)}\right] (X + Y)^{S_b(n)}$$

## Remark

In SageMath, $(-7)_2 = ((-1)(-1)(-1))_2$.

# Explicit expressions $n \in \mathbb{N}$

$$\binom{-n}{k}_b^{(1)} := \prod_{l=0}^{N-1} \binom{-n_l}{k_l}, \quad \binom{-n}{k}_b^{(2)} := \left[x^k\right] \prod_{l=0}^{N-1} \left(1 + x^{b^l}\right)^{-n_l}, \quad \binom{-n}{k}_b^{(3)} := \left[X^{-S_b(n)-S_b(k)} Y^{S_b(k)}\right] (X+Y)^{-S_b(n)}$$

## Proposition (L. J, D. Shi)

$$\binom{-n}{k}_b^{(2)} = \begin{cases} \displaystyle\sum_{(j_0,\ldots,j_{N-1}) \in \mathcal{P}_N(k, \mathrm{b}_N)} \prod_{\ell=0}^{N-1} \binom{-n_\ell}{j_\ell}, & \text{if } k \geq 0; \\ \displaystyle\sum_{(j_0,\ldots,j_{N-1}) \in \mathcal{P}_N^*(-k, \mathrm{b}_N)} \prod_{\ell=0}^{N-1} \binom{-n_\ell}{-j_\ell}, & \text{if } k < 0, \end{cases}$$

$$\binom{-n}{k}_b^{(3)} := \begin{cases} \displaystyle\sum_{j_0+\cdots+j_{N-1} = S_b(k)} \prod_{\ell=0}^{N-1} \binom{-n_\ell}{j_\ell}, & \text{if } k \geq 0; \\ \displaystyle\sum^{*}_{j_0+\cdots+j_{N-1} = -S_b(k)} \prod_{\ell=0}^{N-1} \binom{-n_\ell}{-j_\ell}, & \text{if } k < 0, \end{cases}$$

# Explicit expressions $n \in \mathbb{N}$

$$\sum_{(j_0,\ldots,j_{N-1})\in\mathcal{P}(k,\mathrm{b}_N)}, \quad \sum_{(j_0,\ldots,j_{N-1})\in\mathcal{P}^*(-k,\mathrm{b}_N)}, \quad \sum_{j_0+\cdots+j_{N-1}=S_b(k)}, \quad \sum_{j_0+\cdots+j_{N-1}=-S_b(k)}^{*}$$

1. $\displaystyle\sum_{j_0+\cdots+j_{N-1}=S_b(k)}$ is summing over *nonnegative* integers $j_0,\ldots,j_{N-1}$;
2. $\displaystyle\sum_{j_0+\cdots+j_{N-1}=S_b(k)}^{*}$ is summing over *positive* integers $j_0,j_1,\ldots,j_{N-1}$;
3. $\mathrm{b}_N := \{1,b,\ldots,b^{N-1}\}$ and $\mathcal{P}(k,\mathrm{b}_N)$ is the set of *restricted partitions* of $k$ into parts in $\mathrm{b}_N$ , i.e., $N$-tuples of nonnegative integers $(j_0,\ldots,j_{N-1})$ such that

$$j_0b^0 + j_1b^1 + \cdots + j_{N-1}b^{N-1} = k;$$

4. $\mathcal{P}^*(-k,\mathrm{b}_N) = \mathcal{P}(-k,\mathrm{b}_N)\cap\mathbb{N}^N$ ($N$-tuples with positive component).

## Example: $b=4,\ n=-6,\ |n|=(12)_4,\ f(x)=\frac{1}{(1+x)^2(1+x^4)}$

1. $k=7=(13)_4,\ S_4(k)=4.$
   - $\binom{-6}{7}_4^{(1)}=\binom{-1}{1}\binom{-2}{3}=\frac{(-1)}{1}\cdot\frac{(-2)(-3)(-4)}{3!}=(-1)\cdot(-4)=4;$
   - $f(x)=1-2x+\cdots+4x^6-4x^7+O\left(x^8\right).\Rightarrow\binom{-6}{7}_4^{(2)}=-4;$
   - $(X+Y)^{-3}=X^{-3}\left(1-\cdots+15\frac{Y^4}{X^4}-O\left(\frac{Y^5}{X^5}\right)\right)\Rightarrow\binom{-6}{7}_4^{(3)}=15.$
2. $k=-8,\ 8=(20)_4\ ,\ S_4(-8)=-2$
   - $\binom{-6}{-8}_4^{(1)}=\binom{-1}{-2}\binom{-2}{0}=(-1)\cdot 1=-1;$
   - $f(x)=\frac{1}{x^6}-\frac{2}{x^7}+\frac{3}{x^8}+O\left(\frac{1}{x^9}\right).\Rightarrow\binom{-6}{-8}_4^{(2)}=3;$
   - $(X+Y)^{-3}=X^{-3}\left(1+\frac{Y}{X}\right)^{-3}=X^{-3}\left(\frac{X^3}{Y^3}-\cdots\right)\Rightarrow\binom{-6}{-8}_4^{(3)}=0.$
3. Alternatively, $4_2=\{1,4\}$.
   - $7=1\cdot4+3=0\cdot4+7\Rightarrow\mathcal{P}(7,4_2)=\{(1,3),(0,7)\}.$

$$\binom{-6}{7}_4^{(2)}=\binom{-1}{1}\binom{-2}{3}+\binom{-1}{0}\binom{-2}{7}=(-1)\cdot(-4)+1\cdot(-8)=-4.$$

- $\{(j_0,j_1)\in\mathbb{N}^2:j_0+j_1=2\}=\{(1,1)\}\Rightarrow\binom{-6}{-8}_4^{(3)}=\binom{-1}{-1}\binom{-2}{-1}=1\cdot0=0.$

# Proof and Explanation

$$(1+x)^{-n} = \frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k = \sum_{j=1}^{\infty} \binom{-n}{-j} \frac{1}{x^j}.$$

From the generating function

$$\sum_{k=0}^{\infty} \binom{-n}{k}_b^{(2)} x^k = \prod_{\ell=0}^{N-1} \left( \frac{1}{1+x^{b^\ell}} \right)^{n_\ell} = \prod_{\ell=0}^{N-1} \left( \sum_{j_\ell=0}^{\infty} \binom{-n_\ell}{j_\ell} x^{j_l b^\ell} \right).$$

When, $n > 0$, each factor is $\left(1+x^{b^\ell}\right)^{n_\ell}$, which restricts each $j_\ell \in [0, n_\ell] \subset [0, b-1]$.

$$j_0 b^0 + j_1 b^1 + \cdots + j_{N-1} b^{N-1} = k \Rightarrow j_\ell = k_\ell$$

Thus,

$$\binom{n}{k}_b = \prod_{\ell=0}^{N-1} \binom{n_\ell}{k_\ell} = \left[x^k\right] \prod_{l=0}^{N-1} \left(1+x^{b^\ell}\right)^{n_\ell}.$$

The case for $(X+Y)^{S_b(n)}$ is similar.

# Lucas' theorem

Recall

$$\sum_{k=0}^{n}\binom{n}{k}_p x^k=\prod_{\ell=0}^{N-1}\left(1+x^{p^\ell}\right)^{n_\ell}\equiv(1+x)^n=\sum_{k=0}^{n}\binom{n}{k}x^k \pmod{p}$$

Taking the reciprocal indicates the following.

## Proposition (L. J, D. Shi)

*For $n,k\in\mathbb{Z}$ and a prime $p$,*

$$\binom{n}{k}\equiv\binom{n}{k}_p^{(2)} \pmod{p}.$$

Neither $\binom{n}{k}_p^{(3)}$ nor $\binom{n}{k}_p^{(1)}$ satisfies this congruence.

## Remark

It is obvious that $\binom{n}{k}_p^{(1)}$ has only finitely many non-zero terms, for fixed $n$.

# Pascal's triangle

SageMath

## Conjecture (L. J, D. Shi)

*For the following recurrence*

$$\binom{n}{k}_b^{(j)} + \binom{n}{k-1}_b^{(j)} = \binom{n+1}{k}_b^{(j)},$$

1. *when* $j = 2$*, it holds* $\forall - n, b \in \mathbb{N}$ *and* $\forall k \in \mathbb{Z}$*;*
2. *when* $j = 1$*, it holds if* $k$ *has more digits than* $n$*, or* $k$ *is nonnegative and* $b \nmid k$*, or if* $k$ *is negative and* $k \not\equiv 1 \pmod{b}$*;*
3. *when* $j = 3$*, it hold if* $k$ *is nonnegative and* $b \nmid k$*, or if* $k$ *is negative and* $k \not\equiv 1 \pmod{b}$*.*

# Symmetry

## Conjecture (∼2019-09-05)

$$\binom{n}{k}_b^{(2)} = \binom{n}{n-k}_b^{(2)}$$

*For* $n, k \in \mathbb{N}$,

$$\binom{-n}{k}_b^{(2)} = \binom{-n}{-n-k}_b^{(2)} \quad \textit{and} \quad \binom{-n}{-k}_b^{(2)} = \binom{-n}{-n+k}_b^{(2)}$$

## Proposition (L. J, D. Shi 2019-09-15/16)

$$\binom{n}{k}_b^{(2)} = \binom{n}{n-k}_b^{(2)}.$$

# Symmetry

For $n, k \in \mathbb{N}$,

$$\binom{-n}{k}_b^{(2)} = \binom{-n}{-n-k}_b^{(2)} \quad \text{and} \quad \binom{-n}{-k}_b^{(2)} = \binom{-n}{-n+k}_b^{(2)}$$

$$\sum_{k=0}^{\infty} \binom{-n}{k}_b^{(2)} x^k = \underbrace{\prod_{\ell=0}^{N-1} \left(1 + x^{b^\ell}\right)^{-n_\ell}}_{f(x)=} = \sum_{k={\color{red}n}}^{\infty} \binom{-n}{-k}_b^{(2)} \frac{1}{x^k}.$$

$$f\left(\frac{1}{x}\right) = \prod_{\ell=0}^{N-1} \frac{1}{\left(1 + \left(\frac{1}{x}\right)^{b^\ell}\right)^{n_\ell}} = \prod_{\ell=0}^{N-1} \frac{x^{n_\ell b^\ell}}{\left(x^{b^\ell} + 1\right)^{n_\ell}} = x^n f(x).$$

$$(FG)^{(m)} = \sum_{j=0}^{m} \binom{m}{j} F^{(j)} G^{(m-j)}.$$

# Chu-Vandermonde identity

SageMath

Somehow, it fails..............

# More elementary properties?

As mentioned before, $\binom{n}{k}_b^{(1)}$, for fixed $n$, has only finitely many nonzero values for $k \in \mathbb{Z}$.

## Example

Let $b = 3$. $n = -4$, $4 = (11)_3$.

| $k$ | $-8$ | $-7$ | $-6$ | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | $8$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $1$ | $-1$ | $-1$ | $-1$ | $1$ | $1$ | $-1$ | $1$ | $1$ | $-1$ | $1$ | $-1$ | $1$ | $-1$ | $1$ | $-1$ | $1$ |

What is the sum $\pmod 3$? $= 1$

See SageMath

It seems that for any $n \in \mathbb{N}$,

$$\sum_{k\in\mathbb{Z}} \binom{-n}{k}_3^{(1)} \not\equiv -1 \pmod 3.$$

$$\sum_{k=0}^{n} \binom{n}{k}_3 \equiv \sum_{k=0}^{n} \binom{n}{k} = 2^n \not\equiv 0 \pmod 3.$$

# Restricted partition

## Definition

Given a vector d $:= (d_1, \ldots, d_m)$ of positive integers. Let

$$W(s, \mathrm{d}) = \left|\{(x_1, \ldots, x_n) \in \mathbb{N}^n : d_1x_1 + d_2x_2 + \cdots + d_mx_m = s\}\right|.$$

$$F(t, \mathrm{d}) = \prod_{j=1}^{m} \frac{1}{1 - t^{d_j}} = \sum_{s=0}^{\infty} W(s, \mathrm{d})t^s.$$

$\mathrm{b}_N := \{1, b, \ldots, b^{N-1}\}$ and $\mathcal{P}(k, \mathrm{b}_N)$ is the set of *restricted partitions* of $k$ into parts in $\mathrm{b}_N$, i.e., $N$-tuples of nonnegative integers $j_0, \ldots, j_{N-1}$ such that $j_0b^0 + j_1b^1 + \cdots + j_{N-1}b^{N-1} = k$;

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[1] K. Dilcher and C. Vignat, An explicit form of the polynomial part of a restricted partition function, Research in Number Theory 2017, 3:1.