INTRODUCTION TO FOUR SYMBOLIC INTEGRATION METHODS: TWO EXAMPLES

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1. Introduction

The following two examples will be used.

Example 1.1.

$$I := \int_0^\infty \frac{\mathrm{d}x}{1+x^2} = \tan^{-1}(x)\Big|_{x=0}^{x=\infty} = \frac{\pi}{2}.$$

Example 1.2.

$$I' := \int_0^\infty \frac{\sin x}{x}\mathrm{d}x = \frac{\pi}{2}$$

2. The method of brackets (MoB)

MoB evaluates the definite integral

$$\int_0^\infty f(x)\,dx$$

(most of the time) in terms of SERIES, with *ONLY SEVEN* rules.

Definition 2.1. The *indicator* is defined by

$$\phi_n := \frac{(-1)^n}{n!} = \frac{(-1)^n}{\Gamma(n+1)}.$$

Also, their product is denoted by

$$\phi_{1,\dots,r} := \phi_{n_1,\dots,n_r} = \phi_{n_1}\phi_{n_2}\cdots\phi_{n_r} = \prod_{i=1}^r \phi_{n_i}.$$

Rules

P_1: If

$$f(x) = \sum_{n=0}^\infty a_n x^{\alpha n+\beta-1},$$

then,

$$\int_0^\infty f(x)\,dx \mapsto \sum_n a_n \langle \alpha n+\beta\rangle$$

which is called the bracket series;

P_2: For $\alpha < 0$,

$$(a_1+\cdots+a_r)^\alpha \mapsto \sum_{n_1,\dots,n_r} \phi_{1,\dots,r} a_1^{n_1}\cdots a_r^{n_r} \frac{\langle -\alpha+n_1+\cdots+n_r\rangle}{\Gamma(-\alpha)};$$

P_3: For each bracket series, we assign <u>*index*=# *of sums*− # *of brackets*</u>;

E_1:

$$\sum_n \phi_n F(n) \langle \alpha n + \beta \rangle = \frac{1}{|\alpha|} F(n^*) F(-n^*),$$

where n^* solves $\alpha n + \beta = 0$;

E_2:

$$\sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} F(n_1,\ldots,n_r) \prod_{i=1}^{r} \langle a_{i1}n_1 + \cdots + a_{ir}n_r + c_i \rangle$$

$$= \frac{F(n_1^*,\ldots,n_r^*) \prod_{i=1}^{r} \Gamma(-n_i^*)}{|\det A|},$$

where

$$(n_1^*,\ldots,n_r^*) \text{ solves } \begin{cases} a_{11}n_1 + \cdots + a_{1r}n_r + c_1 & = 0 \\ \ldots & \cdots \\ a_{r1}n_1 + \cdots + a_{rr}n_r + c_r & = 0 \end{cases};$$

E_3: The value of a multi-dimensional bracket series of ***POSITIVE*** index is obtained by computing all the contributions of maximal rank by Rule E_2. These contributions to the integral appear as series in the free parameters. Series converging in a common region are added and divergent series are discarded. Any series producing a non-real contribution is also discarded.

E_4 : Let $k \in \mathbb{N}$ be fixed. In the evaluation of series, the rule

$$(-km)_{-m} = \frac{k}{k+1} \cdot \frac{(-1)^m (km)!}{((k+1)m)!}$$

must be used to eliminate Pochhammer symbols with negative index and negative integer base.

Example 2.2. We let

$$f(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} \phi_n \Gamma(n+1) x^{2n}.$$

(1) By P_1,

$$\int_0^\infty \frac{1}{1+x^2} \mathrm{d}x = \int_0^\infty f(x) \mathrm{d}x = \sum_{n=0}^{\infty} \phi_n \Gamma(n+1) \langle 2n+1 \rangle.$$

(2) To apply E_1, we solve for n^*:

$$2n^* + 1 = 0 \Longrightarrow n^* = -\frac{1}{2}.$$

By E_1,

$$I = \frac{1}{|2|} \Gamma(n^*+1) \Gamma(-n^*) = \frac{1}{2} \Gamma\left(\frac{1}{2}\right)^2 = \frac{1}{2} \left(\sqrt{\pi}\right)^2 = \frac{\pi}{2}.$$

Alternatively, we can first apply P_2 to have

$$\frac{1}{1+x^2} = \left(1+x^2\right)^{-1} = \sum_{n_1,n_2} \phi_{n_1}\phi_{n_2} 1^{n_1} x^{2n_2} \frac{\langle 1+n_1+n_2\rangle}{\Gamma(1)} = \sum_{1,2} \phi_{1,2} x^{2n_2} \langle 1+n_1+n_2\rangle .$$

Then,

$$I = \int_0^\infty \sum_{1,2} \phi_{1,2} x^{2n_2} \langle 1+n_1+n_2\rangle \,\mathrm{d}x = \sum_{1,2} \phi_{1,2} \langle n_1+n_2+1\rangle \langle 2n_2+1\rangle .$$

Solving the linear system:

$$\left.\begin{aligned} n_1+n_2 &= -1 \\ 2n_2 &= -1 \end{aligned}\right\} \Rightarrow \begin{cases} \det \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = 2 \\ (n_1^*, n_2^*) = \left(-\frac{1}{2}, -\frac{1}{2}\right) \end{cases} .$$

By E_2,

$$I = \frac{1}{|2|}\Gamma\left(\frac{1}{2}\right)^2 = \frac{\pi}{2}.$$

Example 2.3. We let

$$f(x) = \frac{\sin x}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n} = \sum_{n=0}^{\infty} \phi_n \frac{\Gamma(n+1)}{\Gamma(2n+2)} \langle 2n+1\rangle .$$

Then,

$$I' = \frac{1}{|2|} \frac{\Gamma\left(-\frac{1}{2}+1\right)}{\Gamma\left(2\left(-\frac{1}{2}\right)+2\right)} \Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)^2 = \frac{\pi}{2}.$$

3. Negative dimension integration method

Example 3.1. Consider for $\alpha > 0$,

$$J(\alpha) := \int_{\mathbb{R}} e^{-\alpha(1+x^2)} \mathrm{d}x.$$

- By the Gaussian distribution,

$$1 = \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \mathrm{d}x.$$

  Let $\mu = 0$ and $1/(2\sigma^2) = \alpha$. We have

$$J(\alpha) = e^{-\alpha} \int_{\mathbb{R}} e^{-\alpha x^2} \mathrm{d}x = e^{-\alpha} \sqrt{\frac{\pi}{\alpha}} = \sqrt{\pi} \sum_{n=0}^{\infty} \phi_n \alpha^{n-\frac{1}{2}}. \tag{3.1}$$

- On the other hand, assuming that we can interchange the integration and summation, we have

$$J(\alpha) = \int_{\mathbb{R}} \sum_{m=0}^{\infty} \phi_m \alpha^m (1+x^2)^m \mathrm{d}x = \sum_{m=0}^{\infty} \phi_m \left(\int_{\mathbb{R}} (1+x^2)^m \mathrm{d}x \right) \alpha^m. \tag{3.2}$$

Now, define

$$I_m := \int_{\mathbb{R}} (1+x^2)^m \mathrm{d}x.$$

To match (3.1) and (3.2), we see $n - 1/2 = m$, i.e., $n = m + 1/2$. Comparing coefficients indicates

$$\begin{aligned}\sqrt{\pi}\phi_{m+\frac{1}{2}} = \phi_m I_m \Longrightarrow I_m &= \sqrt{\pi}\frac{\phi_{m+\frac{1}{2}}}{\phi_m} \\ &= \sqrt{\pi}\frac{(-1)^{m+\frac{1}{2}}}{(-1)^m} \cdot \frac{\Gamma(m+1)}{\Gamma\left(m+\frac{3}{2}\right)} \\ &= \frac{\sqrt{\pi}i}{(m+1)_{\frac{1}{2}}}.\end{aligned}$$

Thus, by

$$(-a)_n = (-1)^n (a-n+1)_n,$$

we have

$$I_m = \frac{\sqrt{\pi}i}{(-1)^{\frac{1}{2}}\left(-m-\frac{1}{2}\right)_{\frac{1}{2}}} = \frac{\sqrt{\pi}}{\left(-m-\frac{1}{2}\right)_{\frac{1}{2}}},$$

so that

$$I = \frac{I_{-1}}{2} = \frac{1}{2} \cdot \frac{\sqrt{\pi}}{\left(\frac{1}{2}\right)_{\frac{1}{2}}} = \frac{1}{2} \cdot \frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = \frac{\pi}{2}.$$

4. Integration by differentiation

Theorem 4.1 (A. Kempf, D. Jackson, and A. Morales)**.** *Let* $\partial_\varepsilon = \partial/\partial\varepsilon$*.*

$$\begin{aligned}\int_a^b f(x)\,dx &= \lim_{\varepsilon\to 0} f(\partial_\varepsilon)\frac{e^{\varepsilon b} - e^{\varepsilon a}}{\varepsilon}, \\ \int_{-\infty}^{\infty} f(x)\,dx &= \lim_{\varepsilon\to 0} 2\pi f(-i\partial_\varepsilon)\,\delta(\varepsilon) = 2\pi\delta(i\partial_\varepsilon) f(\varepsilon), \\ \int_0^{\infty} f(x)\,dx &= \lim_{\varepsilon\to 0} f(-\partial_\varepsilon)\frac{1}{\varepsilon}, \\ \int_{-\infty}^{0} f(x)\,dx &= \lim_{\varepsilon\to 0} f(\partial_\varepsilon)\frac{1}{\varepsilon}, \\ \int_{-\infty}^{\infty} f(x)\,dx &= \lim_{\varepsilon\to 0}\left[f(-\partial_\varepsilon) + f(\partial_\varepsilon)\right]\frac{1}{\varepsilon},\end{aligned}$$

where $i^2 = 1$ *and* δ *is the Dirac-delta function.*

Example 4.2. Rewrite

$$f(x) = \frac{\sin x}{x} = \frac{1}{x} \cdot \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$

$$I' = \lim_{\varepsilon\to 0} f(-\partial_\varepsilon)\frac{1}{\varepsilon} = \frac{1}{2i}\lim_{\varepsilon\to 0}\left(e^{-i\partial_\varepsilon} - e^{i\partial_\varepsilon}\right)\frac{1}{\partial_\varepsilon} \circ \frac{1}{\epsilon}.$$

Note that $1/\partial_\varepsilon$ is the inverse operation of derivative, i.e., integration.

$$I' = \frac{1}{2i}\lim_{\varepsilon\to 0}\left(e^{-i\partial_\varepsilon} - e^{i\partial_\varepsilon}\right) \circ (\ln\varepsilon + c)$$

Recall that for the derivative operator ∂_ε,

$$e^{a\partial_\varepsilon} \circ f(\varepsilon) = \sum_{n=0}^{\infty}\frac{a^n}{n!}\partial_\varepsilon \circ f(\varepsilon) = \sum_{n=0}^{\infty}\frac{a^n}{n!} f^{(n)}(\varepsilon) = f(\varepsilon + a).$$

Therefore,

$$\begin{aligned} I' &= \frac{1}{2i}\lim_{\varepsilon\to 0}\left[(\ln(\varepsilon - i) + c) - (\ln(\varepsilon + i) + c)\right] \\ &= \frac{1}{2i}\lim_{\varepsilon\to 0}\left[\ln(\varepsilon - i) - \ln(\varepsilon + i)\right] = \frac{1}{2i}\left(\frac{-i\pi}{2} - \frac{i\pi}{2}\right) = \frac{\pi}{2}. \end{aligned}$$

5. Hypergeometric form

Proposition 5.1 (P. Blaschke)**.** *Let f be holomorphic near the origin, $\alpha \neq -1$, $\beta \neq 0$ and $-(\alpha+1)/\beta \notin \mathbb{N}$. Then,*

$$\int x^\alpha f(x^\beta)\mathrm{d}x = \frac{x^{\alpha+1}}{\alpha+1} f\left(\begin{matrix} \frac{\alpha+1}{\beta} \\ 1+\frac{\alpha+1}{\beta}\end{matrix} \,\middle|\, x^\beta\right) + C,$$

where

$$f\left(\begin{matrix} a \\ c \end{matrix}\,\middle|\, x\right) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \cdot \frac{f^{(n)}(0)}{n!} x^n.$$

Corollary 5.2. *For $p \leq q+1$,*

$$\int x^\alpha {}_pF_q\left(\begin{matrix} a_1,\dots,a_p \\ c_1,\dots,c_q\end{matrix}\,\middle|\, \gamma x^\beta\right)\mathrm{d}x = \frac{x^{\alpha+1}}{\alpha+1} {}_{p+1}F_{q+1}\left(\begin{matrix} a_1,\dots,a_p, \frac{\alpha+1}{\beta} \\ c_1,\dots,c_q, 1+\frac{\alpha+1}{\beta}\end{matrix}\,\middle|\, \gamma x^\beta\right) + C.$$

Example 5.3. Let

$$f(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n,$$

we see $f^{(n)}(0) = (-1)^n n!$. Thus,

$$\begin{aligned} \int f(x^2)\mathrm{d}x &= x\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{\left(\frac{3}{2}\right)_n} \cdot \frac{(-1)^n n!}{n!} x^{2n} \\ &= x\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n (1)_n}{\left(\frac{3}{2}\right)_n} \cdot \frac{(-x^2)^n}{n!} \\ &= x\,{}_2F_1\left(\begin{matrix} 1, \frac{1}{2} \\ \frac{3}{2}\end{matrix}\,\middle|\, -x^2\right) \\ &= \tan^{-1}(x). \end{aligned}$$

Thus,

$$I = \tan^{-1}(x)\Big|_{x=0}^{x=\infty} = \frac{\pi}{2}.$$

Example 5.4. Note that

$$
\begin{aligned}
{}_0F_1\left(\begin{matrix} - \\ \frac{3}{2} \end{matrix} \,\middle|\, -\frac{x^2}{4}\right) &= \sum_{n=0}^{\infty} \frac{1}{\left(\frac{3}{2}\right)_n} \cdot \frac{(-1)^n x^{2n}}{4^n \cdot n!} \\
&= \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{\left(\frac{3}{2} \cdot \frac{5}{2} \cdots \frac{2n+1}{2}\right) 2^n \cdot 2^n n!} \\
&= \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\
&= \frac{\sin x}{x}.
\end{aligned}
$$

Thus,

$$\int \frac{\sin x}{x} \mathrm{d}x = x \; {}_1F_2\left(\begin{matrix} \frac{1}{2} \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \,\middle|\, -\frac{x^2}{4}\right) (= \mathrm{Si}(x)) \, .$$

Lemma 5.5. *As* $z \to -\infty$,

$$(-z)^{\alpha} {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ c_1, \ldots, c_q \end{matrix} \,\middle|\, z\right) \to \prod_{\substack{l=1 \\ \alpha_l \neq \alpha}}^{p} \frac{\Gamma\left(a_i - \alpha\right)}{a_i} \prod_{j=1}^{q} \frac{\Gamma\left(c_j\right)}{\Gamma\left(c_j - \alpha\right)},$$

iff $p \geq q - 1$ *and*

- *for* $p > q - 1,\ \alpha = \min\left(a_1, \ldots, a_p\right);$
- *for* $p = q - 1,\ \alpha = \min\left(a_1, \ldots, a_p\right) < \sigma - \frac{1}{2}$*,where*

$$\sigma = \sum_j c_j - \sum_l a_l.$$

Let $z = -x^2/4 \Leftrightarrow x = 2(-z)^{1/2}$, we see

$$\alpha = \frac{1}{2} = \min\left(\frac{1}{2}\right) < 2 = \left(\frac{3}{2} + \frac{3}{2} - \frac{1}{2}\right) - \frac{1}{2}.$$

Thus,

$$\lim_{x \to \infty} x \; {}_1F_2\left(\begin{matrix} \frac{1}{2} \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \,\middle|\, -\frac{x^2}{4}\right) = 2\left(\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(1\right)}\right)^2 = \frac{\pi}{2}.$$

On the other hand,

$$\lim_{x \to 0} x \; {}_1F_2\left(\begin{matrix} \frac{1}{2} \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \,\middle|\, -\frac{x^2}{4}\right) = 0 \cdot 1 = 0.$$

Therefore,

$$I' = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

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