# Hankel Determinants on Sequences Related to Bernoulli and Euler Polynomials

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# Hankel Determinants

## Definition

Given a sequence $\mathrm{a} = (a_k)_{k=0}^{\infty}$, the $n$-th *Hankel determinant* of a is defined by

$$H_n(\mathrm{a}) := \det_{0 \le i,j \le n} (a_{i+j}) = \det \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{pmatrix}.$$

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## Example

Let b $= (B_n)_{n=0}^{\infty}$ be the sequence of Bernoulli numbers, determined by the generating function

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. Then, the first few terms of $H_n(\mathrm{b})$ are

$$1, -\frac{1}{12}, -\frac{1}{540}, \frac{1}{42000}, \frac{1}{3215625}, -\frac{4}{623959875}, -\frac{64}{213746467935}.$$

# Orthogonal Polynomials

Suppose we are given a sequence $c = (c_0, c_1, ...)$ of numbers; then it is known that there exists a positive Borel measure $\mu$ on $\mathbb{R}$ with infinite support such that

$$c_k = \int_{\mathbb{R}} y^k d\mu(y), \qquad k = 0, 1, 2, \ldots$$

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we have

$$H_n(\mathrm{c}) = t_1^n t_2^{n-1} \cdots t_{n-1}^2 t_n.$$

# Continued Fractions

## Theorem

*Let* $c_0 \neq 0$*.*

$$\sum_{k=0}^{\infty} c_k z^k = \cfrac{c_0}{1 - s_0 z - \cfrac{t_1 z^2}{1 - s_1 z - \cfrac{t_2 z^2}{1 - s_2 z - \ddots}}}$$

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$$H_n(B_k(x)) = H_n(B_k) = (-1)^{\binom{n+1}{2}} \prod_{\ell=1}^{n} \left( \frac{\ell^4}{4(2\ell+1)(2\ell-1)} \right)^{n+1-\ell}.$$

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## Definition

Euler polynomials $E_n(x)$ and Euler numbers $E_n$ are defined by

$$\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \qquad \frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

$$H_n(b_k) = (-1)^{\varepsilon(n)} a^{n+1} \prod_{\ell=1}^{n} b(\ell)^{n+1-\ell}$$

| $b_k$ | $\varepsilon(n)$ | $a$ | $b(\ell)$ |
|---|---|---|---|
| $B_k$ | $\binom{n+1}{2}$ | $1$ | $\frac{\ell^4}{4(2\ell+1)(2\ell-1)}$ |
| $B_{k+1}$ | $\binom{n+2}{2}$ | $\frac{1}{2}$ | $\frac{\ell^2(\ell+1)^2}{4(2\ell+1)^2}$ |
| $B_{k+2}$ | $\binom{n+1}{2}$ | $\frac{1}{6}$ | $\frac{\ell(\ell+1)^2(\ell+2)}{4(2\ell+1)(2\ell+3)}$ |
| $B_{2k+2}$ | $0$ | $\frac{1}{6}$ | $\frac{\ell^3(\ell+1)(2\ell-1)(2\ell+1)^3}{(4\ell-1)(4\ell+1)^2(4\ell+3)}$ |
| $B_{2k+4}$ | $n+1$ | $\frac{1}{30}$ | $\frac{\ell(\ell+1)^3(2\ell+1)^3(2\ell+3)}{(4\ell+1)(4\ell+3)^2(4\ell+5)}$ |
| $B_{2k}(\frac{1}{2})$ | $0$ | $1$ | $\frac{\ell^4(2\ell-1)^4}{(4\ell-3)(4\ell-1)^2(4\ell+1)}$ |
| $(2^{2k+2}-1)B_{2k+2}$ | $0$ | $\frac{1}{2}$ | $\ell^3(\ell+1)$ |
| $(2k+1)B_{2k}(\frac{1}{2})$ | $0$ | $1$ | $\frac{\ell^6}{4(2\ell+1)(2\ell-1)}$ |
| $(2k+3)B_{2k+2}$ | $0$ | $\frac{1}{2}$ | $\frac{\ell^3(\ell+1)^3}{4(2\ell+1)^2}$ |
| $B_{2k+1}(\frac{x+1}{2})$ | $\binom{n+1}{2}$ | $\frac{x}{2}$ | $\frac{\ell^4(x^2-\ell^2)}{4(2\ell+1)(2\ell-1)}$ |
| $E_k$ | $\binom{n+1}{2}$ | $1$ | $\ell^2$ |
| $E_k(x)$ | $\binom{n+1}{2}$ | $1$ | $\frac{\ell^2}{4}$ |
| $E_{k+1}(1)$ | $\binom{n+1}{2}$ | $\frac{1}{2}$ | $\frac{\ell(\ell+1)}{4}$ |

| | | | |
|---|---|---|---|
| $E_{2k}$ | $0$ | $1$ | $(2\ell-1)^2(2\ell)^2$ |
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| $E_{2k+2}$ | $n+1$ | $1$ | $(2\ell)^2(2\ell+1)^2$ |
| $E_{2k+3}(1)$ | $n+1$ | $\frac{1}{4}$ | $\frac{\ell(\ell+1)(2\ell+1)^2}{4}$ |
| $(2k+1)E_{2k}$ | $0$ | $1$ | $(2\ell)^4$ |
| $(2k+2)E_{2k+1}(1)$ | $0$ | $1$ | $\ell^3(\ell+1)$ |
| $\frac{E_{k+1}(1)}{(k+1)!}$ | $\binom{n+1}{2}$ | $\frac{1}{2}$ | $\frac{1}{4(2\ell-1)(2\ell+1)}$ |
| $\frac{E_{2k+1}(1)}{(2k+1)!}$ | $0$ | $\frac{1}{2}$ | $\frac{1}{16(4\ell-3)(4\ell-1)^2(4\ell+1)}$ |
| $\frac{E_{2k+3}(1)}{(2k+3)!}$ | $n+1$ | $\frac{1}{24}$ | $\frac{1}{16(4\ell-1)(4\ell+1)^2(4\ell+3)}$ |

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| $E_{2k}(\frac{x+1}{2})$ | $\binom{n+1}{2}$ | $1$ | $\frac{\ell^2}{4}(x^2-(2\ell-1)^2)$ |
| $E_{2k+1}(\frac{x+1}{2})$ | $\binom{n+1}{2}$ | $\frac{x}{2}$ | $\frac{\ell^2}{4}(x^2-(2\ell)^2)$ |
| $E_{2k+2}(\frac{x+1}{2})$ | $\binom{n+1}{2}$ | $\frac{x^2-1}{4}$ | $\frac{\ell^2}{4}(x^2-(2\ell+1)^2)$ |

# Real Results (K. Dilcher & L. J 2020)

| $b_k$ | $\varepsilon(n)$ | $a$ | $b(\ell)$ |
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| $\left(2^{2k+2}-1\right)B_{2k+2}$ | $0$ | $\frac{1}{2}$ | $\ell^3(\ell+1)$ |
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| $(2k+3)B_{2k+2}$ | $0$ | $\frac{1}{2}$ | $\frac{\ell^3(\ell+1)^3}{4(2\ell+1)^2}$ |
| $(2k+1)E_{2k}$ | $0$ | $1$ | $16\ell^4$ |
| $(2k+2)E_{2k+1}(1)$ | $0$ | $1$ | $\ell^3(\ell+1)$ |
| $B_{2k+1}\left(\frac{x+1}{2}\right)$ | $\binom{n+1}{2}$ | $\frac{x}{2}$ | $\frac{\ell^4(x^2-\ell^2)}{4(2\ell+1)(2\ell-1)}$ |
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# Motivation

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| $n$ | $H_n\left(B_{2k+1}\left(\frac{x+1}{2}\right)\right)$ |
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| 1 | $\frac{1}{2}x$ |
| 2 | $-\frac{1}{48}x^2(x^2-1)$ |
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2. $H_n(a_k)$ and $H_n(a_{2k})$ are totally different, unless $a_{2k+1} \equiv 0$.
3. Let $\chi$ be a primitive Dirichlet character $\bmod f$. The generalized Bernoulli numbers and polynomials belonging to $\chi$ are defined by

$$\sum_{a=1}^{f} \frac{\chi(a)te^{at}}{e^{ft}-1} = \sum_{n=0}^{\infty} B_{n,\chi}\frac{t^n}{n!} \quad \text{and} \quad B_{n,\chi}(x) = \sum_{k=0}^{n} \binom{n}{k} B_{k,\chi}x^{n-k}.$$

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In particular, $\boxed{B_{n+1,\chi_4} = -\frac{n+1}{2}E_n}$, where $\chi_4$ is the unique non-trivial character with $f=4$, i.e., $\chi_4(1)=1$, $\chi_4(3)=-1$, and $\chi_4(2)=\chi_4(4)=0$.

# Characters

## Theorem (K. Dilcher and L. J. 2020)

*For* $q = 4$ *or* $6$, *let*

$$b_k^{(j)} = \frac{1}{k+1} B_{k+1,\chi_{2q,j}}(x), \quad j = 1, 2,$$

*where*

| $n$ | 1 | 3 | 5 | 7 |
|---|---|---|---|---|
| $\chi_{8,1}$ | 1 | $-1$ | $-1$ | 1 |
| $\chi_{8,2}$ | 1 | 1 | $-1$ | $-1$ |

*and*

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$$H_{2m}(b_k^{(1)}) = 0 \qquad (\tilde{q} = (q-2)/q)$$

$$H_{2m+1}(b_k^{(1)}) = (-1)^{m+1} \left( \frac{q-2}{2} q^{2m} \right)^{2m+2} \prod_{\ell=1}^{m} \left( \frac{\ell^2}{4} (\tilde{q}^2 - (2\ell)^2 \right)^{2(m+1-\ell)}$$

$$H_{2m}(b_k^{(2)}) = (-1)^{m+1} \frac{q^{2m(2m+1)}}{m!^2} \prod_{\ell=1}^{m} \left( \frac{\ell^2}{4} (\tilde{q}^2 - (2\ell-1)^2 \right)^{2(m+1-\ell)}$$

$$H_{2m+1}(b_k^{(2)}) = \left( \frac{q^{2m+1}}{2} \right)^{2m+2} \prod_{\ell=0}^{m} \frac{\ell!^4}{16^\ell} \left( \frac{\ell^2}{4} (\tilde{q}^2 - (2\ell+1)^2 \right)^{2(m+1-\ell)}$$

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- Given a $= (a_0, a_1, \ldots) \leftrightarrow P_n$, there is a formula to compute $H_n(a_{k+1})$. But when adding a term at the very beginning, i.e., $(c, a_0, a_1, \ldots)$, things are getting complicated.

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## Conjecture (K. Dilcher and L. J.)

$$H_n((2k+1)B_{2k}) = (-1)^n \prod_{\ell=1}^{n} \left( \frac{\ell^6}{4(2\ell+1)(2\ell-1)} \right)^{n+1-\ell} \cdot (H(n) + H(n+1))$$

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## Definition

Unfortunately, we have to denote

$$H(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

## Conjecture (K. Dilcher and L. J.)

$$H_n((2k{+}1)B_{2k}) = (-1)^n \prod_{\ell=1}^{n} \left( \frac{\ell^6}{4(2\ell+1)(2\ell-1)} \right)^{n+1-\ell} \cdot (H(n) + H(n+1))$$

## Theorem (K. Dilcher and L. J.)

$$H_n((2k+3)B_{2k+2}) = \frac{1}{2^{n+1}} \prod_{\ell=1}^{n} \left( \frac{\ell^3(\ell+1)^3}{4(2\ell+1)^2} \right)^{n+1-\ell}.$$

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Thank You!!