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学术交流会

DUKE KUNSHAN
Zu Chongzhi Center for Mathematics and Computational Sciences

Hankel Determinants of Sequences related to Bernoulli and Euler Polynomials

Lin Jiu

Joint work with Karl Dilcher

WHU-DKU May 27th, 2021

Karl Dilcher

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Chapter 24 Bernoulli and Euler Polynomials

K. Dilcher
Dalhousie University, Halifax, Nova Scotia, Canada.


Hankel Determinant

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Given a sequence a $= (a_0, a_1, \ldots)$, the nth Hankel determinant of a is defined by

$$H_n(\mathrm{a}) = H_n(a_k) := \det_{0 \le i,j \le n} \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{pmatrix}$$

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- H_n is the determinant of an $n+1$ by $n+1$ matrix (Hankel matrix);
- It is important to begin with $k = 0$ for a.

Main Results: $H_n(a_k)$ for the following sequences

$$B_{2k+1}\left(\frac{x+1}{2}\right), E_{2k}\left(\frac{x+1}{2}\right), E_{2k+1}\left(\frac{x+1}{2}\right), E_{2k+2}\left(\frac{x+1}{2}\right),$$

$$B_k\left(\frac{x+r}{q}\right) - B_k\left(\frac{x+s}{q}\right), E_k\left(\frac{x+r}{q}\right) \pm E_k\left(\frac{x+s}{q}\right),$$
$$kE_{k-1}(x), B_{k+1,x_{8,1}}(x), B_{k+1,x_{8,2}}(x), B_{k+1,x_{12,1}}(x), B_{k+1,x_{12,2}}(x),$$
$$(2k+1)E_{2k}, (2^{2k+2}-1)B_{2k+2}, (2k+1)B_{2k}\left(\frac{1}{2}\right), (2k+3)B_{2k+2},$$

$a_k,\ k \geq 1$	B_{k-1}	B_{2k}	$(2k+1)B_{2k}$	$(2^{2k}-1)B_{2k}$
a_0	0	1	1	0

$a_k,\ k \geq 1$	E_{2k-2}	$E_{k-1}(1)$	$E_{k+3}(1)$	$E_{2k-1}(1)$	$E_{2k+5}(1)$	$\frac{E_k(1)}{k!}$
a_0	0	0	$-\frac{1}{4}$	0	$\frac{1}{2}$	1

$a_k,\ k \geq 1$	$\frac{E_{2k-1}(1)}{(2k-1)!}$	$E_{2k-2}\left(\frac{x+1}{2}\right)$	$(2k+1)E_{2k}$		
a_0	0	0	0		

Main Results: References

1. arXiv:2105.01880 [pdf, ps, other] math.NT

Hankel Determinants of shifted sequences of Bernoulli and Euler numbers

Authors: Karl Dilcher, Lin Jiu

Abstract: Hankel determinants of sequences related to Bernoulli and Euler numbers have been studied before, and numerous identities are known. However, when a sequence is shifted by one unit, the situation often changes significantly. In this paper we use classical orthogonal polynomials and related methods to prove a general result concerning Hankel determinants for shifted sequences. We then apply this re... ▽ More

Submitted 5 May, 2021; **originally announced** May 2021.

MSC Class: Primary 11B68; Secondary 33D45; 11C20

2. arXiv:2007.09821 [pdf, ps, other] math.NT math.CO

Hankel Determinants of sequences related to Bernoulli and Euler Polynomials

Authors: Karl Dilcher, Lin Jiu

Abstract: We evaluate the Hankel determinants of various sequences related to Bernoulli and Euler numbers and special values of the corresponding polynomials. Some of these results arise as special cases of Hankel determinants of certain sums and differences of Bernoulli and Euler polynomials, while others are consequences of a method that uses the derivatives of Bernoulli and Euler polynomials. We also obt... ▽ More

Submitted 19 July, 2020; **originally announced** July 2020.

MSC Class: Primary 11B68; Secondary 11C20

3. arXiv:2006.15236 [pdf, ps, other] math.NT math.CA math.CV

Orthogonal polynomials and Hankel Determinants for certain Bernoulli and Euler Polynomials

Authors: Karl Dilcher, Lin Jiu

Abstract: Using continued fraction expansions of certain polygamma functions as a main tool, we find orthogonal polynomials with respect to the odd-index Bernoulli polynomials $B_{2k+1}(x)$ and the Euler polynomials $E_{2k+\nu}(x)$, for $\nu = 0, 1, 2$. In the process we also determine the corresponding Jacobi continued fractions (or J-fractions) and Hankel determinants. In all these cases the Hankel determinants... ▽ More

Submitted 26 June, 2020; **originally announced** June 2020.

Motivation: Basic Facts on H_n

1. Orthogonal Polynomials and Continued Fractions

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iff $H_n(a_k) \neq 0$, for $n = 0, 1, 2, \ldots$, there exist the unique monic orthogonal polynomial sequence $P_n(y)$ such that

- deg $P_n = n$;
- and $\mathcal{L}(P_n(x)P_m(y)) = \zeta_n \delta_{m,n}$;

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iff $H_n(a_k) \neq 0$, for $n = 0, 1, 2, \dots,$ there exist the unique monic orthogonal polynomial sequence $P_n(y)$ such that

- deg $P_n = n$;
- and $\mathcal{L}(P_n(x)P_m(y)) = \zeta_n \delta_{m,n}$;

Fact. $P_n(y)$ satisfies a 3-term recurrence

$$P_{n+1}(y) = (y + s_n)P_n(y) - t_n P_{n-1}(y),$$

with $P_0 = 1$, $P_{-1} = 0$, for some sequences s_n and t_n. Then,

$$H_n(a_k) = a_0^{n+1} t_1^n t_2^{n-1} \cdots t_n.$$

Orthogonal Polynomials and Continued Fractions

THM.

$$\sum_{n=0}^{\infty} a_n z^n = \cfrac{a_0}{1 + s_0 z - \cfrac{t_1 z^2}{1+s_1 z - \cfrac{t_2 z^2}{1+s_2 z - \ddots}}}$$

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   The Bernoulli numbers/polynomials are defined by

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1} \quad \text{and} \quad \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{e^{xt} t}{e^t - 1}.$$

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$$B_{2n} + \sum_{(p-1)|2n} \frac{1}{p} \in \mathbb{Z}.$$

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while

$$H_6(B_k) = \det \begin{pmatrix} B_0 & \cdots & B_6 \\ \vdots & \ddots & \vdots \\ B_6 & \cdots & B_{12} \end{pmatrix} = -\frac{64}{3 \cdot 5 \cdot 7 \cdot 11 \cdot 13}$$

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$$H_n(B_k) = (-1)^{\frac{n(n+1)}{2}} \prod_{\ell=1}^{n} \left(\frac{\ell^4}{4(2\ell+1)(2\ell-1)} \right)^{n+1-\ell}.$$

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n	$H_n\left(B_{2k+1}\left(\frac{x+1}{2}\right)\right)$
0	$\frac{x}{2}$
1	$-\frac{1}{48}x^2(x^2-1)$
2	$-\frac{x^3(x^2-1)^2(x^2-2^2)}{4320}$
3	$\frac{x^4(x^2-1)^3(x^2-2^2)^2(x^2-3^2)}{672000}$

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3. Searching for New Methods

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Lem.

$$\sum_{k=0}^{\infty} B_{2k+1}\left(\frac{x+1}{2}\right) z^{2k} = \frac{1}{2z^2}\left[\psi'\left(\frac{1}{z}+\frac{1-x}{2}\right) - \psi'\left(\frac{1}{z}+\frac{1+x}{2}\right)\right]$$

$$= \cfrac{\frac{x}{2}}{1+\sigma_0 z^2 - \cfrac{\tau_1 z^4}{1+\sigma_1 z^2 - \cfrac{\tau_2 z^4}{1+\sigma_2 z^2 - \ddots}}},$$

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$$\psi'(x) = \frac{d^2}{dx^2}\left(\log \Gamma(x)\right),$$

$$\sigma_n = \binom{n+1}{2} - \frac{x^2-1}{4},$$

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The continued fraction can be found at, e.g.,

`B. C. Berndt, Ramanujan's Notebooks, Part II . Springer-Verlag,`

Know Approaches

1. Orthogonal Polynomials and Continued Fractions

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$$B_{2k+1}\left(\frac{x+1}{2}\right), E_{2k}\left(\frac{x+1}{2}\right), E_{2k+1}\left(\frac{x+1}{2}\right), E_{2k+2}\left(\frac{x+1}{2}\right)$$

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Euler polynomials

$$\sum_{n=0}^{\infty} E_k(x)\frac{t^n}{n!} = \frac{2e^{xt}}{e^t+1} \quad \text{and} \quad \sum_{n=0}^{\infty} E_k\frac{t^n}{n!} = \frac{2}{e^t+e^{-t}}.$$

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For $\nu = 0, 1, 2$, the continued fraction expressions of

$$\sum_{k=0}^{\infty} E_{2k+\nu}\left(\frac{x+1}{2}\right) z^{2k}$$

can also be found in the literature, related to $\psi(x) = [\log \Gamma(x)]'$.

Remark. We actually spent almost a month to find those expressions.

2. Left-shifted Sequence

$$H_n(b_{k+1}) = H_n(b_k) \cdot \det \begin{pmatrix} -s_0 & 1 & 0 & 0 & \cdots & 0 \\ t_1 & -s_1 & 1 & 0 & \cdots & 0 \\ 0 & t_2 & -s_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & t_n & -s_n \end{pmatrix},$$

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Ex.

$$B_{2k+1}\left(\frac{x+1}{2}\right), E_{2k}\left(\frac{x+1}{2}\right), E_{2k+1}\left(\frac{x+1}{2}\right), E_{2k+2}\left(\frac{x+1}{2}\right)$$

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- "checkerboard" determinants

$$\det\begin{pmatrix} a & 0 & b & 0 & c \\ 0 & d & 0 & e & 0 \\ f & 0 & g & 0 & h \\ 0 & i & 0 & j & 0 \\ k & 0 & l & 0 & m \end{pmatrix} = \det\begin{pmatrix} a & b & c \\ f & g & h \\ k & l & m \end{pmatrix} \cdot \det\begin{pmatrix} d & e \\ i & j \end{pmatrix}$$

by J. Cigler and C. Krattenthaler for general determinants.

Some New Methods

1. Derivatives:

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Lem. Let $c_k(x)$ be a sequence of C^1 functions, and let $P_n(y;x)$ be the corresponding monic orthogonal polynomials. If $c_k(x_0) = 0$ for some $x_0 \in \mathbb{C}$ and for all $k \geq 0$, then $P_n(y;x_0)$ are the monic orthogonal polynomials with respect to the sequence of derivatives $c_k'(x_0)$, as long as $H_n(c_k'(x_0))$ are all nonzero.

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Ex. $(2k+1)E_{2k}$

$$E_{2k+1}\left(\frac{x+1}{2}\right) \longleftrightarrow P_{n+1} = \left(y + (2n+1)\left(n+\frac{1}{2}\right) - \frac{x^2-1}{4}\right)P_n - \frac{n^2(x^2-4n^2)}{4}P_{n-1};$$

and recall that

$$E_k'(x) = kE_{k-1}(x) \quad \text{and} \quad E_{2k+1}\left(\frac{1}{2}\right) = 0.$$

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Given c_k, we define

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Namely,

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Ex.

$$H_n(B_{2k}) = (-1)^n \frac{(4n+3)!}{(n+1) \cdot (2n+1)!^3} H_n(B_{2k+2}) \mathcal{H}_{2n+1},$$

for the harmonic numbers

$$\mathcal{H}_n = 1 + \frac{1}{2} + \cdots \frac{1}{n}.$$

Right-shifted

Lem. Let s_n and t_n be the sequences appearing in the 3-term recurrence of the monic orthogonal polynomial sequences for c_k. Then,

$$\frac{H_{n+1}(b_k)}{H_n(c_k)} = -s_n \frac{H_n(b_k)}{H_{n-1}(c_k)} - t_n \frac{H_{n-1}(b_k)}{H_{n-2}(c_k)}.$$

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$a_k,\ k \geq 1$	B_{k-1}	B_{2k}	$(2k+1)B_{2k}$	$(2^{2k}-1)B_{2k}$
a_0	0	1	1	0

$a_k,\ k \geq 1$	E_{2k-2}	$E_{k-1}(1)$	$E_{k+3}(1)$	$E_{2k-1}(1)$	$E_{2k+5}(1)$	$\frac{E_k(1)}{k!}$
a_0	0	0	$-\frac{1}{4}$	0	$\frac{1}{2}$	1

$a_k,\ k \geq 1$	$\frac{E_{2k-1}(1)}{(2k-1)!}$	$E_{2k-2}\left(\frac{x+1}{2}\right)$	$(2k+1)E_{2k}$		
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Further Remark

Let us take Bernoulli numbers B_k and Bernoulli polynomials $B_k(x)$ as an example.

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1. We know the orthogonal polynomials w. r. t. B_k;
2. $H_n(B_k(x)) = H_n(B_k)$
3. The left-shifted sequences $H_n(B_{k+1})$ and $H_n(B_{k+2})$ are known.

Further Remark

Let us take Bernoulli numbers B_k and Bernoulli polynomials $B_k(x)$ as an example.
1. We know the orthogonal polynomials w. r. t. B_k;
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DUKE KUNSHAN
Zu Chongzhi Center for Mathematics and Computational Sciences

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We still do not know how to compute the Hankel determinants of some arbitrary sequence.

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