EXAMPLES ON COMPUTER PROOFS

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Example 1

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n\ (n+1)\ (2\ n+1)}{6}$$

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Out[ ]= $1 - 3 n + 3 n^2$

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$$\vdots$$

$$1^3 - 0^3 = 3\cdot 1^2 - 3\cdot 1 + 1$$

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$$n^3 = 3 \sum_{k=1}^{n} k^2 - 3 \sum_{k=1}^{n} k + n$$

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Check that

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In[•]:= f[n_] := Sum[k^2, {k, 1, n}];
        g[n_] := n (n + 1) (2 n + 1) / 6;

In[•]:= Table[f[n] - g[n], {n, 1, 4}]

Out[•]= {0, 0, 0, 0}

In[•]:= f[3] - g[3]

Out[•]= 0
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WZ – Method

$R(n, k) := 1/2 \times (2k - 3n - 3)\, k^2 / ((k - n - 1)^2 (2n + 1))$

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F[n_, k_] := Binomial[n, k]^2 / Binomial[2 n, n];
R[n_, k_] := 1 / 2 * (2 * k - 3 * n - 3) * k^2 / ((k - n - 1)^2 * (2 * n + 1));
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In[•]:= F[n + 1, k] - F[n, k] - (G[n, k + 1] - G[n, k]) // FullSimplify

Out[•]= 0
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$$\sum_{k=-\infty}^{\infty} [F(n+1, k) - F(n, k)] = \lim_{k \to \infty} G(n, k+1) - \lim_{k \to -\infty} G(n, k)$$

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In[ ]:=
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F[n + 1, k] - F[n, k] - (G[n, k + 1] - G[n, k]) // FullSimplify
```

Out[ ]=
```
0
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$$\sum_{k=-\infty}^{\infty} [F(n+1, k) - F(n, k)] = \lim_{k \to \infty} G(n, k+1) - \lim_{k \to -\infty} G(n, k) = 0.$$

$$\sum_{k=0}^{n} F(n, k) = \sum_{k=-\infty}^{\infty} F(n, k) = \sum_{k=-\infty}^{\infty} F(0, k) = 1.$$

In[∘]:= $BaseDirectory

Out[∘]= C:\ProgramData\Mathematica

In[∘]:= << RISC`GeneratingFunctions`

Package GeneratingFunctions version 0.8 written by Christian Mallinger
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

In[∘]:= RE2DE[
{a[n + 1] - (8 n^2 + 8 n + 3) a[n] + (2 n)^4 * a[n - 1] == 0, a[0] == 0, a[1] == 1}, a[n], f[x]]

Out[∘]= $\{-x + (1 - 3x + 16x^2) f[x] + 16(-x^2 + 15x^3) f'[x] + 8(-x^3 + 50x^4) f''[x] + 160x^5 f^{(3)}[x] + 16x^6 f^{(4)}[x] == 0, f[0] == 0, f'[0] == 1, f''[0] == 38, f^{(3)}[0] == 4278\}$

In[∘]:= RE2DE[{a[n] + (n + 1) * a[n + 1] - (n + 2) * a[n + 2] == 0, a[0] == 1, a[1] == 0}, f[x]]

In[∘]:= RE2DE[{a[n] + (1 + n) a[1 + n] - (2 + n) a[2 + n] == 0, a[0] == 1, a[1] == 0}, a[n], f[x]]

Out[∘]= $\{x f[x] + (-1 + x) f'[x] == 0, f[0] == 1\}$

In[∘]:= RE2DE[{a[n + 2] - a[n + 1] - a[n] == 0, a[0] == 1, a[1] == 1}, a[n], f[x]]

Out[∘]= $-1 - (-1 + x + x^2) f[x] == 0$

In[∘]:= DSolve[{x f[x] + (-1 + x) f'[x] == 0, f[0] == 1}, f, x]

Out[∘]= $\{\{f \to \text{Function}[\{x\}, -\frac{e^{-x}}{-1 + x}]\}\}$

In[∘]:= ? DSolve

Out[∘]= Symbol

DSolve[*eqn*, *u*, *x*] solves a differential equation for the function *u*, with independent variable *x*.

DSolve[*eqn*, *u*, {*x*, x_{min}, x_{max}}] solves a differential equation for *x* between x_{min} and x_{max}.

DSolve[{eqn_1, eqn_2, ...}, {u_1, u_2, ...}, ...] solves a list of differential equations.

DSolve[*eqn*, *u*, {x_1, x_2, ...}] solves a partial differential equation.

DSolve[*eqn*, *u*, {x_1, x_2, ...} ∈ Ω] solves the partial differential equation *eqn* over the region Ω.

In[∘]:= << RISC`FASTZEIL`

Fast Zeilberger Package version 3.61
written by Peter Paule, Markus Schorn, and Axel Riese
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

In[∘]:=
```
? Zb
```

Symbol

Zb[function, range, n, order],

uses Zeilberger's algorithm to find a recurrence relation of given order in n

for the sum of the function over the range.

Zb[function, k, n, order],

uses Zeilberger's algorithm to find a recurrence relation of given order in n

for the function. This recurrence is — up to a telescoping part —

free of k.

In both calls, if the order is of the form {ord1, ord2}, Zb tries to find

a recurrence whose order is between ord1 and ord2. Omitting the order is equivalent to

specifying {0, Infinity}.

Out[∘]=

In[∘]:=
```
Zb[Binomial[n, k]^2, {k, 0, n}, n, 1]
If `n' is a natural number, then:
```

Out[∘]= {-2 × (1 + 2 n) SUM[n] + (1 + n) SUM[1 + n] == 0}

In[∘]:= **World Population**

(no interpretations available)

Final Remark

- Book A=B: https://www2.math.upenn.edu/~wilf/AeqB.pdf
- RISC Packaged: https://risc.jku.at/software/
- SageMath Website: https://www.sagemath.org/
- Manuel Kauers Website: http://www.kauers.de/software.html