Random Walk and Combinatorial Identities

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Motivation

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Definition

The **Euler numbers** E_n, **Euler polynomials** $E_n(x)$, and **Euler polynomials of order** p $E_n^{(p)}(x)$, are defined via their (exponential) generating functions

$$\frac{2e^t}{e^{2t}+1}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!},\quad \frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},\quad \left(\frac{2}{e^t+1}\right)^p e^{xt}=\sum_{n=0}^{\infty}E_n^{(p)}(x)\frac{t^n}{n!}.$$

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$E_n^{(1)}(x) = E_n(x)$ and $E_n = 2^n E_n(1/2)$.

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Fact: Convolution

$$E_n^{(p)}(x) = \sum_{k_1+\cdots+k_p+k=n} \binom{n}{k_1, \ldots, k_p, k} x^k E_{k_1}(0) E_{k_2}(0) \cdots E_{k_p}(0).$$

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$$E_n(x) = P\left(E_{n_1}^{(p_1)}(x), \ldots, E_{n_k}^{(p_k)}(x)\right)?$$

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For any positive integer N,

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Namely, T_n is the Chebyshev polynomial of the 1st kind.

Acknowledgment

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$\underline{N = 2:}$ $T_2(z) = 2z^2 - 1$ and $\frac{1}{T_2(1/z)} = \frac{z^2}{2-z^2}$

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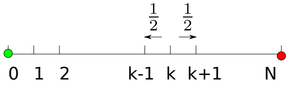

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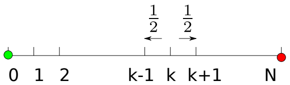

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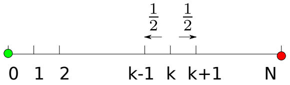

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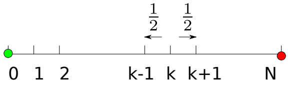

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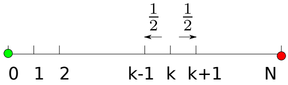

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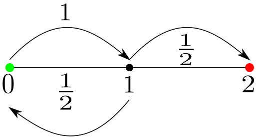

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Theorem (Klebanov et al.)

The random variable

$$Z_N = \frac{1}{N}\sum_{j=1}^{\nu_N} L_j$$

has the same hyperbolic secant distribution (as L_j's).

L. B. Klebanov, A. V. Kakosyan, S. T. Rachev, and G. Temnov. On a class of distributions stable under random summation. *J. Appl. Prob.*, **49** (2012), 303–318.

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Taking moments:

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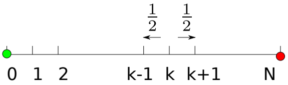
- RHS: Each $\nu_N = \ell$, with probability $p_\ell^{(N)}$ and
$$\mathbb{E}\left[\left(i\sum_{j=1}^{\ell} L_j - \frac{\ell}{2} + Nx - \frac{N}{2} + \frac{\ell}{2}\right)^n\right] = E_n^{(\ell)}\left(\frac{\ell - N}{2} + Nx\right).$$

□

Reflected Brownian Motion



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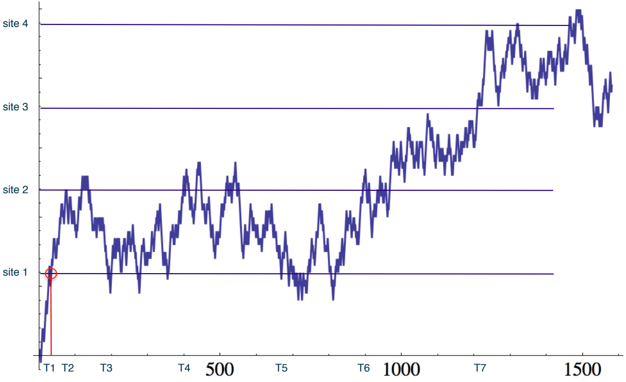

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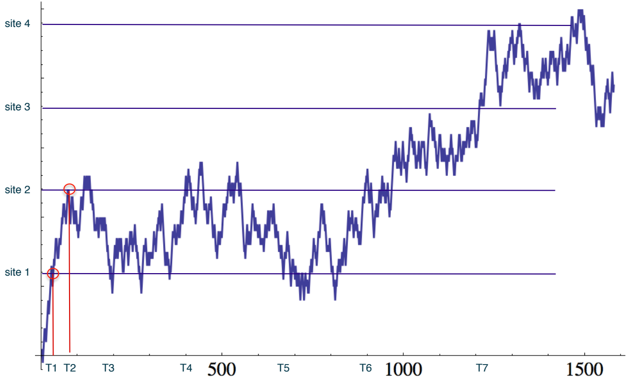

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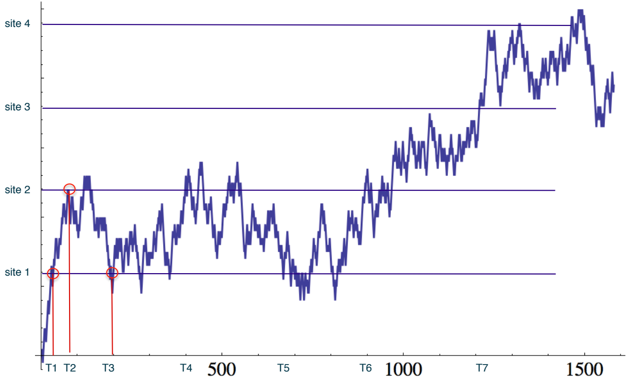

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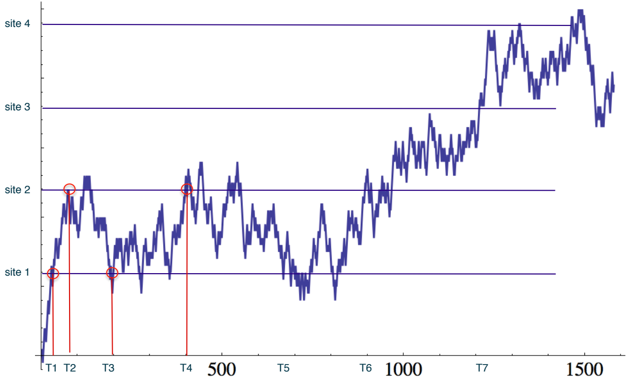

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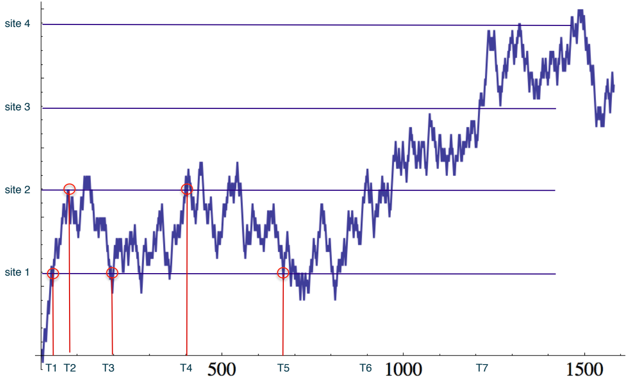

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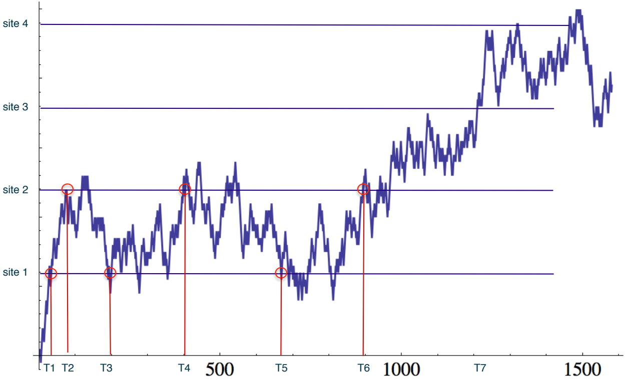

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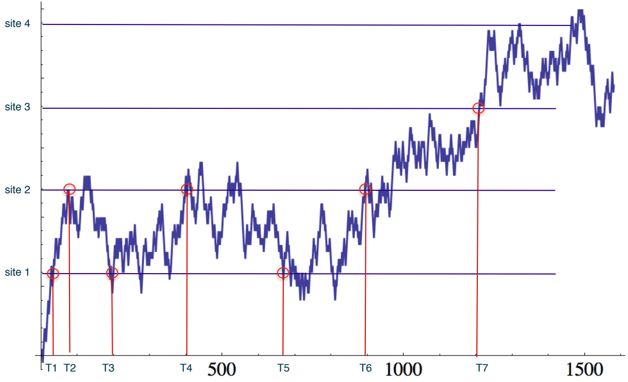

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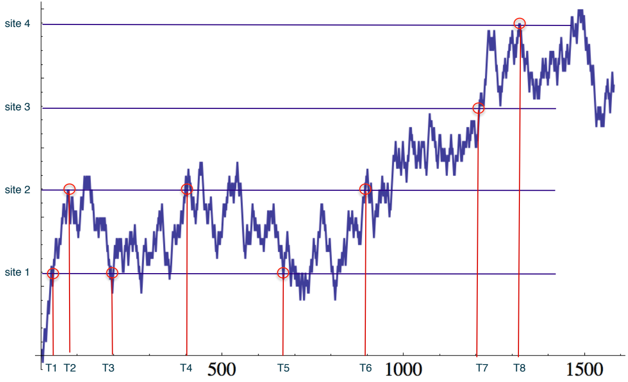

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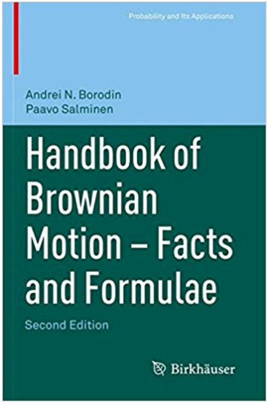

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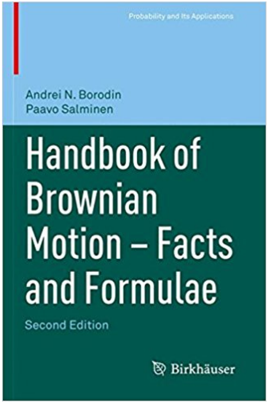

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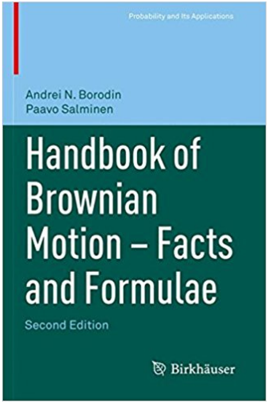

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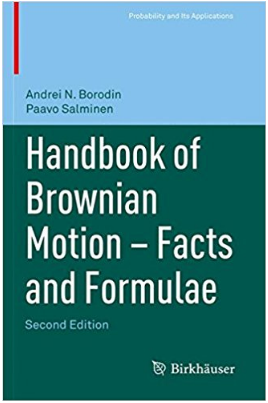

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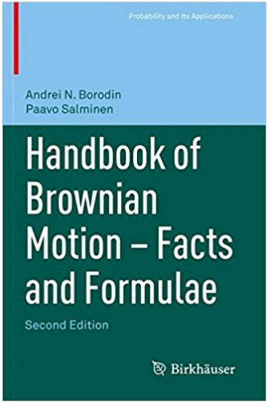

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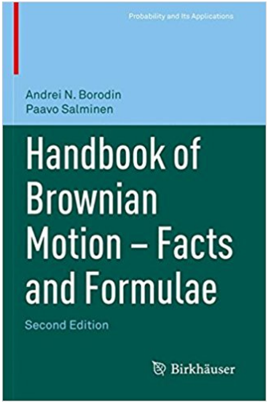

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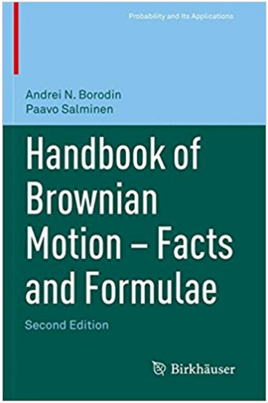

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$$\frac{2}{1+e^s}e^{sx} = \sum_{n=0}^{\infty} E_n(x)\frac{s^n}{n!}$$

Christophe's Idea

Consider a linear Brownian motion W_t starting from 0, with the hitting time T by W_t of level $z = 1$. Define another independent Brownian motion $\omega_t \sim \operatorname{sech}(x)$. Let

$$T_1 < T_2 < \cdots < T_l = T, \quad T_j = \min_s \left\{ W_t = \frac{j}{N} \right\}.$$

This defines a random walk with

$$p_\ell^{(N)} = \mathbb{P}\left\{ W_t \text{ reach the sink in } \ell \text{ steps} \right\}.$$

Now write

$$T = (T - T_{\ell-1}) + (T_{\ell-1} - T_{\ell-2}) + \cdots + (T_1 - 0)$$

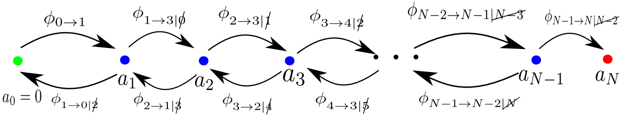
and

$$\omega_T \sim \omega_{T - T_{\ell-1}} + \omega_{T_{\ell-1} - T_{\ell-2}} + \cdots + \omega_{T_1 - 0},$$

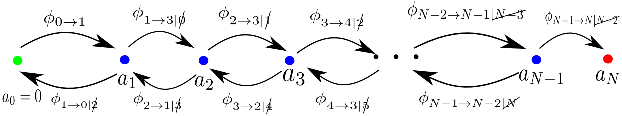
each term $\sim \operatorname{sech}(x)$. This corresponds Klebanov's random sum decomposition

$$Z_N = \frac{1}{N} \sum_{j=1}^{\nu_N} L_j \sim L.$$

My Goal


$\phi_{0\to1}$
$\phi_{1\to3|\not0}$
$\phi_{2\to3|\not1}$
$\phi_{3\to4|\not2}$
$\phi_{N-2\to N-1|\not N-3}$
$\phi_{N-1\to N|\not N-2}$
$a_0=0$
a_1
a_2
a_3
a_{N-1}
a_N
$\phi_{1\to0|\not2}$
$\phi_{2\to1|\not3}$
$\phi_{3\to2|\not4}$
$\phi_{4\to3|\not5}$
$\phi_{N-1\to N-2|\not N}$

My Goal



- A reflected Brownian motion model with N equally distributed sites; $a_j = j$ or j/N;

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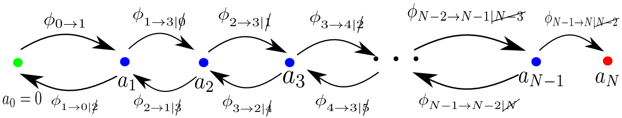

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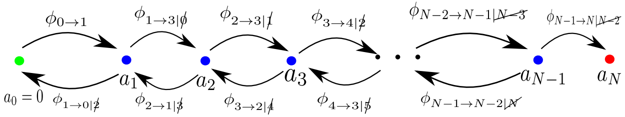

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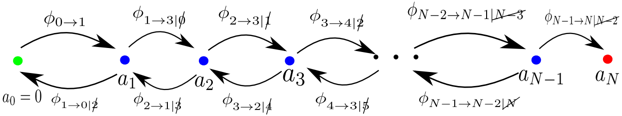
$$\mathbb{E}_x\left[e^{-\alpha H_z}\right] = \begin{cases} \dfrac{\cosh(xw)}{\cosh(zw)}, & 0 \leq x \leq z; \\ e^{-(x-z)w}, & z \leq x. \end{cases}$$

My Goal



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If anyone has an idea, please let me know.

1-dim, 1-loop

With $p \leq q \leq r$, $w = \sqrt{2\alpha}$

$$\phi_{p\to q} := \mathbb{E}_p\left[e^{-\alpha H_q}\right] = \frac{\cosh(pw)}{\cosh(qw)},$$

$$\phi_{q\to p|\not r} := \mathbb{E}_q\left[e^{-\alpha H_p} | W_t < r\right] = \frac{\sinh((r-q)w)}{\sinh((r-p)w)},$$

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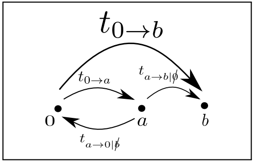

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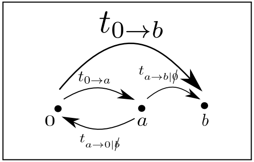

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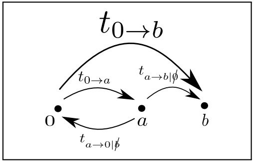

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$$t_{0\to b} = \underbrace{\left(t_{0\to a} + t_{a\to 0|\not{b}}\right) + \cdots + \left(t_{0\to a} + t_{a\to 0|\not{b}}\right)}_{\ell \text{ copies}} + t_{0\to a} + t_{a\to b|\not{0}}$$

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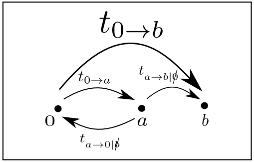

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- Generating functions:

$$\phi_{0\to b} = \phi_{0\to a}\phi_{a\to b|\not 0}\sum_{\ell=0}^{\infty}\left(\phi_{0\to a}\phi_{a\to 0|\not b}\right)^{\ell}$$

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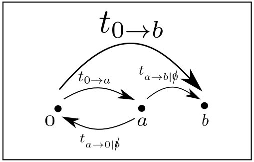

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$$\begin{aligned}\operatorname{sech}(bw) &= \operatorname{sech}(aw)\cdot\frac{\sinh(aw)}{\sinh(bw)}\sum_{\ell=0}^{\infty}\left[\operatorname{sech}(aw)\cdot\frac{\sinh((b-a)w)}{\sinh(bw)}\right]^{\ell}\\ &= \operatorname{sech}(aw)\cdot\frac{\sinh(aw)}{\sinh(bw)}\cdot\frac{1}{1-\operatorname{sech}(aw)\cdot\frac{\sinh((b-a)w)}{\sinh(bw)}}\end{aligned}$$

1-dim, 1-loop

Prop. (L.J. and C. Vignat)

$$E_n\left(\frac{x}{2b}+\frac{3}{2}-2\frac{a}{b}\right)-E_n\left(\frac{x}{b}+\frac{1}{2}\right)=\frac{(n+1)\left(1-2\frac{a}{b}\right)2^n a^n}{b^n}\sum_{\ell=0}^{\infty}\underline{\frac{a}{b}\left(1-\frac{a}{b}\right)^{\ell}}B_n^{(\ell+1)}\left(\frac{x+b}{4a}+\frac{\ell}{2}\right).$$

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- $\frac{a}{b}\left(1-\frac{a}{b}\right)^{\ell}$ are the probability weights of a geometric distribution with parameter a/b.

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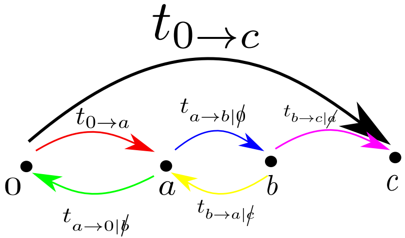
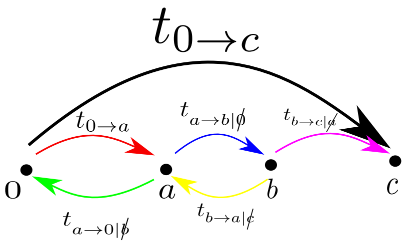
How about 2-loops?

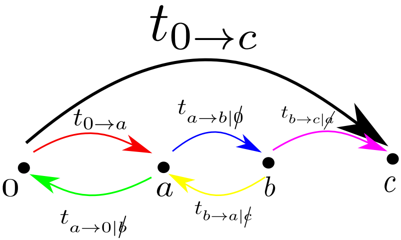


t, t, t, t, t, t

1-dim, 2-loops

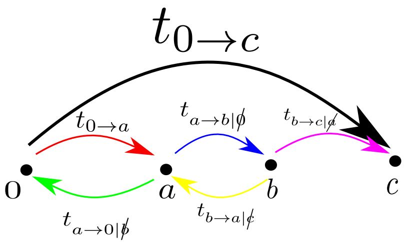

$t_{0\to c}$
$t_{0\to a}$
$t_{a\to b|\not{0}}$
$t_{b\to c|\not{a}}$
0
a
b
c
$t_{a\to 0|\not{b}}$
$t_{b\to a|\not{c}}$

1-dim, 2-loops



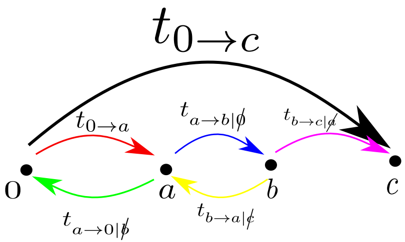
$$t \quad =$$

1-dim, 2-loops



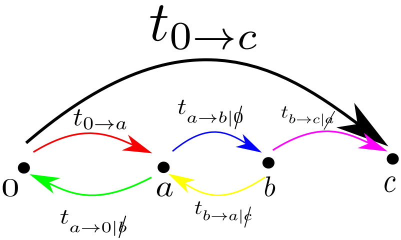
$t \;=\; t$

1-dim, 2-loops



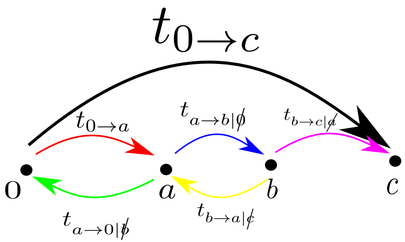
$$t = t + t$$

1-dim, 2-loops



$$t = t + t + t$$

1-dim, 2-loops



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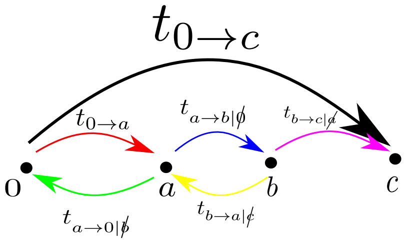
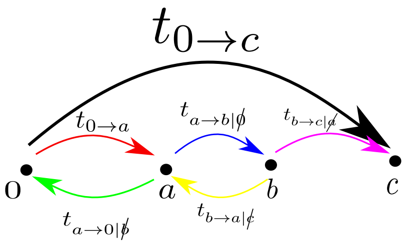
1-dim, 2-loops



$$t = t + t + t + t + t$$

1-dim, 2-loops



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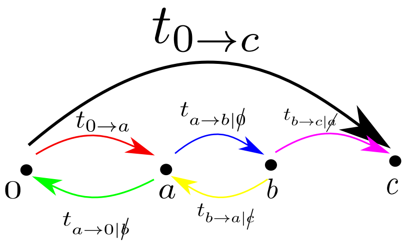
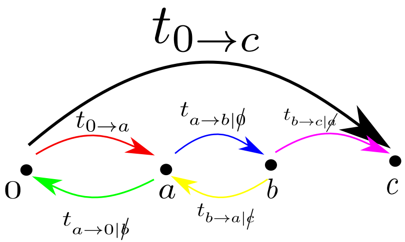


$$t \quad = \quad t + t + t + t + t + t + \cdots$$

1-dim, 2-loops



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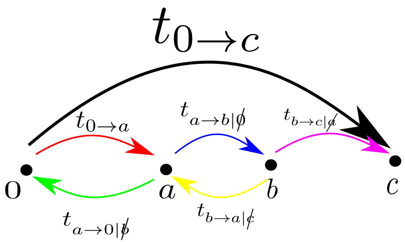
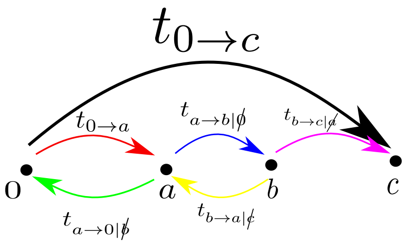
1-dim, 2-loops



$$
\begin{aligned}
t &= t + t + t + t + t + t + \cdots + t + t \\
&= t + t + t + \underbrace{(t+t) + \cdots + (t+t)}_{k \text{ loops}} + \underbrace{(t+t) + \cdots + (t+t)}_{\ell \text{ loops}}
\end{aligned}
$$

1-dim, 2-loops



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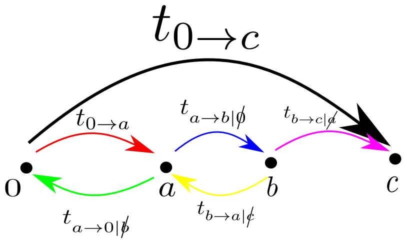
We can generalize it to n-loop model.

1-dim, 2-loops



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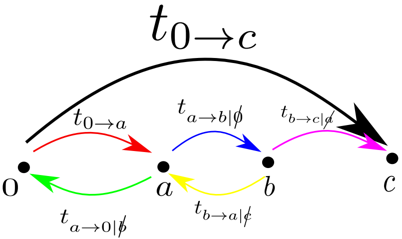
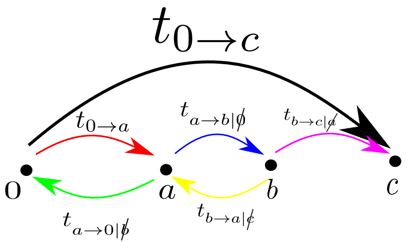
Unfortunately, this is WRONG.....

1-dim, 2-loops



$$\phi = \phi\cdot\phi\cdot\phi\cdot\left[\sum_{k=0}^{\infty}(\phi\phi)^k\right]\left[\sum_{\ell=0}^{\infty}(\phi\phi)^{\ell}\right] = \frac{\phi\cdot\phi\cdot\phi}{(1-\phi\phi)(1-\phi\phi)}$$

1-dim, 2-loops



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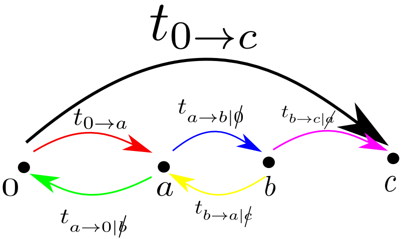
Let $a = 1$, $b = 2$ and $c = 3$.

1-dim, 2-loops



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Let $a=1$, $b=2$ and $c=3$.

$$\text{LHS} = \phi = \phi_{\mathbf{0}\to\mathbf{3}} = \operatorname{sech}(3w) = \frac{1}{\cosh(3w)}$$

$$\text{RHS} = \frac{\frac{1}{\cosh(w)}\cdot\frac{\sinh(w)}{\sinh(2w)}\cdot\frac{\sinh(w)}{\sinh(2w)}}{\left(1-\frac{1}{\cosh(w)}\frac{\sinh(w)}{\sinh(2w)}\right)\left(1-\frac{\sinh(w)}{\sinh(2w)}\frac{\sinh(w)}{\sinh(2w)}\right)} = \frac{\frac{1}{4\cosh^3 w}}{\left(1-\frac{1}{2\cosh^2 w}\right)\left(1-\frac{1}{4\cosh^2 w}\right)}$$

$$= \frac{2\cosh w}{(2\cosh^2 w-1)(4\cosh^2 w-1)} \neq \frac{1}{\cosh(3w)} = \frac{1}{4\cosh^3 w - 3\cosh w}$$

Two-loops

$$I := \phi_{a\to b|\not{A}}\phi_{b\to a|\not{c}}, \quad II := \phi_{b\to c|\not{a}}\phi_{c\to b|\not{B}}$$

- k loops of I followed by l loops of II, with $k, l = 0, 1, \ldots,$ which gives

$$\sum_{k,l} I^k II^l = \frac{1}{1-I} \cdot \frac{1}{1-II};$$

- k_1 loops of I followed by l_1 loops of II, then followed by k_2 loops of I and finally followed by l_2 loops of II, with k_1, l_2 nonnegative and k_2, l_1 positive, which gives

$$\sum_{k_1, l_2=0, k_2, l_1=1}^{\infty} I^{k_1} II^{l_1} I^{k_2} II^{l_2} = \frac{I \cdot II}{(1-I)^2 (1-II)^2};$$

- the general term will be k_1 loops of $I \to l_1$ loops of $II \to \cdots \to k_n$ loops of $I \to l_n$ loops of II, with k_1, l_n nonnegtive and the rest indices positive, which gives

$$\frac{(I \cdot II)^{n-1}}{(1-I)^n (1-II)^n}.$$

Therefore, loops I and II contribute as

$$\sum_{n=1}^{\infty} \frac{(I \cdot II)^{n-1}}{(1-I)^n (1-II)^n} = \frac{1}{1-(I+II)} = \sum_{k=0}^{\infty} (I+II)^k .$$

Two Loops

Prop. (LJ. and C. Vignat)

For any positive integer n,

$$E_n\left(\frac{x}{6}\right) = \sum_{k=0}^{\infty} \frac{3^{k-n}}{4^{k+1}} E_n^{(2k+3)}\left(\frac{x}{2}+k\right).$$

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In general

$$(x+2c\mathcal{E}+c)^n=\sum_{k=0}^{\infty}\sum_{\ell=0}^{k}q_{k,\ell}\Big[x+2(b-a)\mathcal{B}+2(c-b)\mathcal{B}'+a\mathcal{E}^{(\ell)}+2(b-a)\mathcal{U}^{(l)}$$
$$+2a\mathcal{U}'^{(k-\ell)}+2(b-a)\mathcal{B}'^{(k-\ell)}+q'_{k,\ell}\Big]^n,$$

$$q_{k,\ell}:=\binom{k}{\ell}\frac{(b-a)^{\ell+1}a^{k-\ell+1}(c-b)^{k-\ell}}{b^{k+1}(c-a)^{k-\ell+1}}\qquad q'_{k,\ell}=c+(2k-2\ell)b+(3\ell-k+1)a,$$

where

$$\left(\mathcal{E}^{(p)}+x\right)^n=E_n^{(p)}(x),\quad\left(\mathcal{B}^{(p)}+x\right)^n=B_n^{(p)}(x),\quad\mathcal{U}^n=\frac{1}{n+1},\quad\mathcal{U}^{(p)}=\mathcal{U}_1+\cdots+\mathcal{U}_p.$$

n loops?

Consider consecutive loops l_1, l_2, ..., l_n, it seems like the contribution is

$$\sum_{k=0}^{\infty}\left(\sum_{\ell=1}^{n} l_\ell\right)^k = \frac{1}{1-(l_1+\cdots+l_n)}. \quad (*)$$

- It feels right.

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$$\frac{1}{\cosh(Nw)} \overset{??}{=} \frac{\frac{1}{\cosh w}\cdot\left(\frac{\sinh w}{\sinh(2w)}\right)^N}{1-\left(\frac{1}{\cosh(w)}\frac{\sinh(w)}{\sinh(2w)}+(N-1)\frac{\sinh(w)}{\sinh(2w)}\frac{\sinh(w)}{\sinh(2w)}\right)}$$

$$= \frac{\frac{1}{2^N\cosh^{N+1} w}}{1-\frac{N+3}{4}\cosh^N w}.$$

This shows $(*)$ is not correct.

$$\frac{1}{\cosh(Nw)} = \frac{1}{\cos(Niw)} = \frac{1}{T_N(\cos(iw))} = \frac{1}{T_N(\cosh w)}.$$

Acknowledgment

General Formula

Thm. (LJ, I. Simonelli, and H. Yue)

$$\phi_{0\to a_n} = \phi_{0\to a_1}\phi_{a_1\to a_2|\emptyset}\cdots\phi_{a_{n-1}\to a_n|\cancel{a_{n-2}}}\cdot\frac{1}{1-P(L_1,\ldots,L_n)},$$

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for the condition $*$ given by

- $\ell = 1, 2, \ldots, n$;
- $j_1 < j_2 - 1,\ j_2 < j_3 - 1, \ldots, j_{\ell-1} < j_\ell - 1.$

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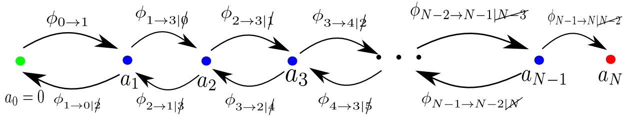

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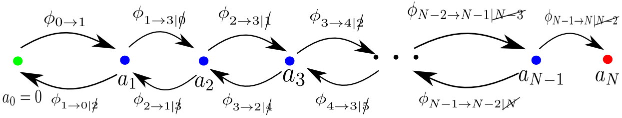

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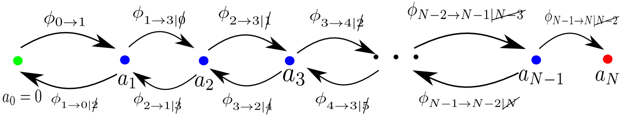

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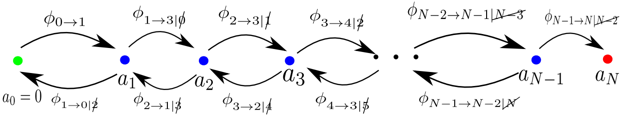

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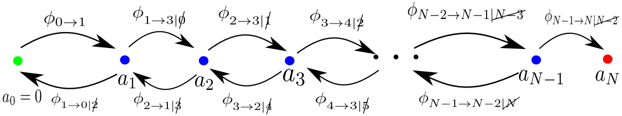

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Results

Thm. (LJ, I. Simonelli, and H. Yue)

By letting $a_j = j$, we have

$$E_n(x) = \frac{1}{4^n}\sum_{k=0}^{\infty}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\frac{2^n}{8^{\ell+1}}E_n^{(2k+2\ell)}(4x+k+\ell).$$

$$E_n(x) = \sum_{k=0}^{\infty}\frac{5^{k-n}}{4^{k+\ell+2}}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}E_n^{(2\ell+2k+5)}(5x+\ell+k).$$

$$\cdots$$

Generalization

- Bessel process in $\mathbb{R}^n$:

$$R_t^{(n)} := \sqrt{\left(W_t^{(1)}\right)^2 + \cdots + \left(W_t^{(n)}\right)^2}$$

- Moment generating functions for hitting times: $H_z := \min_s \left\{ R_s^{(n)} = z \right\}$.

$$\mathbb{E}_x \left(e^{-\alpha H_z}; \sup_{0 \leq s \leq H_z} R_s^{(n)} < y \right) = \begin{cases} \frac{x^{-\nu} I_\nu(xw)}{z^{-\nu} I_\nu(zw)}, & 0 \leq x \leq z \leq y; \\ \frac{S_\nu(yw, xw)}{S_\nu(yw, zw)}, & z \leq x \leq y, \end{cases}$$

- $n = 2 + 2\nu$ for $\nu \geq 0$

$$S_\nu(x, y) := (xy)^{-\nu} \left[I_\nu(x) K_\nu(y) - K_\nu(x) I_\nu(y) \right],$$

and

$$I_\nu(x) = \sum_{\ell=0}^{\infty} \frac{1}{\ell! \Gamma(\ell + \nu + 1)} \left(\frac{x}{2} \right)^{2\ell + \nu}, \quad K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\nu \pi)}.$$

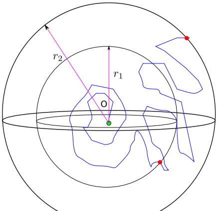
$n = 3 \Leftrightarrow \nu = 1/2$

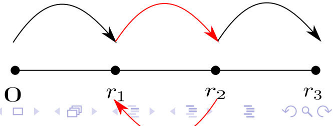
- $$I_{\frac{1}{2}}(x) = \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2m+\frac{1}{2}}}{m!\Gamma\left(m+\frac{3}{2}\right)} = \sqrt{\frac{2}{x\pi}} \sinh(x)$$

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t - 1}e^{tx} = \frac{te^{tx}e^{-\frac{t}{2}}}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} = \frac{te^{t\left(x-\frac{1}{2}\right)}}{2} \sinh\left(\frac{t}{2}\right)$$

$$K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}$$

$$\mathbb{E}_x\left(e^{-\alpha H_z}; \sup_{0\leq s\leq H_z} R_s^{(3)} < y\right) = \begin{cases} \frac{z\sinh(xw)}{x\sinh(zw)}, & 0 \leq x \leq z \leq y \\ \frac{z\sinh((y-x)w)}{x\sinh((y-z)w)}, & z \leq x \leq y \end{cases}$$





$n = 3 \Leftrightarrow \nu = 1/2, r_1 = 1, r_2 = 2, r_3 = 3$

Prop. (LJ. and C. Vignat)

$$\frac{3^{n+1}}{n+1}\left[B_{n+1}\left(\frac{x}{6}+\frac{5}{6}\right)-B_{n+1}\left(\frac{x}{6}+\frac{1}{2}\right)\right]=\sum_{k\geq 0}\frac{3}{4}\left(\frac{1}{4}\right)^{k}E_n^{(2k+2)}\left(\frac{x+3+2k}{2}\right).$$

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Corollary

1. *Take* $x=0$, $n=2m-1$ *in* $(\circledast)$.

$$B_{2m}=\frac{m}{\left(1-2^{1-2m}\right)\left(3^{2m}-1\right)}\sum_{k\geq 0}\left(\frac{1}{4}\right)^{k}E_{2m-1}^{(2k+2)}\left(k+\frac{3}{2}\right).$$

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Prop. (LJ and C. Vignat)

For any positive integer n,

$$3^n B_n\left(\frac{x+4}{6}\right) = \sum_{k=0}^{\infty}\frac{1}{2^k}E_n^{(2k+2)}\left(\frac{x+2k+3}{2}\right).$$

Acknowledgment

Harmonic sums

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$$S_{i_1,\ldots,i_k}(N) = \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{\operatorname{sign}(i_1)^{n_1}}{n_1^{|i_1|}} \times \cdots \times \frac{\operatorname{sign}(i_k)^{n_k}}{n_k^{|i_k|}}$$

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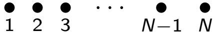

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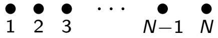
$$\mathbb{P}(6\to 6)=\cdots=\mathbb{P}(6\to 1)=\frac{1}{6}.$$

Walk



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$$\begin{matrix} \bullet & \bullet & \bullet & \cdots & \bullet & \bullet \\ 1 & 2 & 3 & & N-1 & N \end{matrix}$$

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Moments-Cumulants

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Moments-Cumulants

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Thm. (Faà di Bruno's Formula)

$$m_n = Y_n(\kappa_1, \ldots, \kappa_n)$$
$$\kappa_n = \sum_{k=1}^{n} (-1)^k Y_{n,k}(m_1, \ldots, m_{n-k}),$$

where Y_n and $Y_{n,k}$ are the complete and incomplete Bell polynomials, respective.

Thm. (M. Hoffman)

$$Y_k\left(\frac{B_2\cdot 1!}{2\cdot 2!},\frac{B_4\cdot 2!}{4\cdot 4!},\ldots,\frac{B_{2k}\cdot k!}{2k\cdot(2k)!}\right)=\frac{k!}{2^{2k}(2k+1)!}$$

Thm. (B. Y. Rubinstein)

$$Y_k\left(-\frac{B_2\cdot 1!}{2\cdot 2!},-\frac{B_4\cdot 2!}{4\cdot 4!},\ldots,-\frac{B_{2k}\cdot k!}{2k\cdot(2k)!}\right)=\frac{k!(2^{1-2k}-1)B_{2k}}{(2k)!}$$

Prop. (LJ and D. Y. H. Shi)

Moments	Cumulants
$\bar{m}_n = B_n\left(\frac{1}{2}\right)$	$\bar{\kappa}_n = \begin{cases} -B_n/n, & \text{if } n > 1; \\ 0, & \text{if } n = 1. \end{cases}$
$\tilde{m}_n = \begin{cases} \frac{1}{2^n(n+1)}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$	$\tilde{\kappa}_n = -\bar{\kappa}_n = \begin{cases} B_n/n, & \text{if } n > 1; \\ 0, & \text{if } n = 1. \end{cases}$
$m'_n = E_n$	$\kappa'_n = \begin{cases} 2^n(1-2^n)B_n/n & \text{if } n > 1; \\ 0, & \text{if } n = 1. \end{cases}$
$m''_n := \begin{cases} 1, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$	$\kappa''_n = -\kappa'_n = \begin{cases} 2^n(2^n-1)B_n/n & \text{if } n > 1; \\ 0, & \text{if } n = 1. \end{cases}$

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- The "counterparts" are

$$\begin{aligned} B_n &= -n\sum_{\ell=1}^{n}(-1)^{\ell-1}(\ell-1)!Y_{n,\ell}\left(B_1\left(\frac{1}{2}\right),\ldots,B_{n-\ell+1}\left(\frac{1}{2}\right)\right) \\ &= n\sum_{\ell=1}^{n}(-1)^{\ell-1}(\ell-1)!Y_{n,\ell}\left(0,\frac{1}{4\cdot 3},0,\ldots,\frac{1+(-1)^{n-\ell+1}}{2^{n-\ell+2}(n-\ell+2)}\right) \end{aligned}$$

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- Simplification is also necessary, to reduce the degree of the polynomials.

Thank you for your attention