Random Walk Models for Non-trivial Identities Involving Bernoulli and Euler Polynomials of Higher-orders

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@ 2nd Annual Workshop Suzhou Area Youth Mathematicians

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and Computational Sciences

Sept. 25, 2021

Objects

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	Numbers	Polynomials	Polynomials of Order p
Bernoulli	B_n	$B_n(x)$	$B_n^{(p)}(x)$
Euler	E_n	$E_n(x)$	$E_n^{(p)}(x)$

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$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!} \qquad \frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!} \qquad \left(\frac{t}{e^t-1}\right)^p e^{xt}=\sum_{n=0}^{\infty}B_n^p(x)\frac{t^n}{n!}$$

$$\frac{2e^t}{e^{2t}+1}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!} \qquad \frac{2te^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!} \qquad \left(\frac{2t}{e^t+1}\right)^p e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}$$

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$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \qquad \frac{t e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \qquad \left(\frac{t}{e^t - 1}\right)^p e^{xt} = \sum_{n=0}^{\infty} B_n^p(x) \frac{t^n}{n!}$$

$$\frac{2e^t}{e^{2t} + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \qquad \frac{2t e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \qquad \left(\frac{2t}{e^t + 1}\right)^p e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}$$

$$B_n^{(1)}(x) = B_n(x) \qquad B_n(0) = B_n$$

$$E_n^{(1)}(x) = E_n(x) \qquad 2^n E_n\left(\frac{1}{2}\right) = E_n$$

Example

$$B_{n+1}\left(\frac{x+2}{5}\right)-B_{n+1}\left(\frac{x}{5}\right)=\frac{n+1}{5^n}\sum_{k=0}^{\infty}\frac{3^k}{2^{2k+3}}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{\ell}\frac{1}{12^{\ell}}E_n^{(2k+2\ell+3)}(k+\ell+x).$$

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- $\boxed{\tau=e^{\frac{t}{5}}}$

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{1}{n+1}\left(B_{n+1}\left(\frac{x+2}{5}\right)-B_{n+1}\left(\frac{x}{5}\right)\right)\frac{t^n}{n!}\\
&=\frac{1}{t}\sum_{n=0}^{\infty}\left(B_{n+1}\left(\frac{x+2}{5}\right)-B_{n+1}\left(\frac{x}{5}\right)\right)\frac{t^{n+1}}{(n+1)!}\\
&=\frac{1}{t}\left(\left(\frac{te^{\frac{x+2}{5}t}}{e^t-1}-1\right)-\left(\frac{te^{\frac{x}{5}t}}{e^t-1}-1\right)\right)\\
&=\frac{e^{\frac{xt}{5}}}{e^t-1}\left(e^{\frac{2}{5}t}-1\right)=\frac{\tau^x(\tau+1)}{\tau^4+\tau^3+\tau^2+\tau+1}.\end{aligned}$$

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$$=\sum_{k=0}^{\infty}\frac{3^k}{2^{2k+3}}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^\ell\frac{1}{12^\ell}\sum_{n=0}^{\infty}E_n^{(2k+2\ell+3)}(k+\ell+x)\frac{\left(\frac{t}{5}\right)^n}{n!}$$

$$=\sum_{k=0}^{\infty}\frac{3^k}{2^{2k+3}}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^\ell\frac{1}{12^\ell}\left(\frac{2}{\tau+1}\right)^{2k+2\ell+3}\tau^{k+\ell+x}$$

$$=\frac{\tau^x}{2^3}\left(\frac{2}{\tau+1}\right)^3\sum_{k=0}^{\infty}\left(\frac{3}{4}\right)^k\left(\frac{2}{\tau+1}\right)^{2k}\tau^k\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^\ell\frac{1}{12^\ell}\left(\frac{2}{\tau+1}\right)^{2\ell}\tau^\ell$$

$$=\tau^x\left(\frac{1}{\tau+1}\right)^3\frac{1}{1-\frac{3\tau}{(\tau+1)^2}\cdot\frac{3(\tau+1)^2-\tau}{3(\tau+1)^2}}$$

$$=\tau^x\frac{\tau+1}{(\tau+1)^4-\tau\left(3(\tau+1)^2-\tau\right)}.$$

First Example

Dr. Victor H. Moll

Dr. Christophe Vignat

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$$\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = \frac{2}{e^z+1}e^{xt}$$

$$\sum_{n=0}^{\infty} E_n^{(p)}(x)\frac{t^n}{n!} = \left(\frac{2}{e^z+1}\right)^p e^{xt}$$

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$$\left.\begin{aligned} \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} &= \frac{2}{e^z+1}e^{xt} \\ \sum_{n=0}^{\infty} E_n^{(p)}(x)\frac{t^n}{n!} &= \left(\frac{2}{e^z+1}\right)^p e^{xt} \end{aligned}\right\} \Rightarrow E_n^{(p)}(x) = \sum_{k_1+\cdots+k_p=n} \binom{n}{k_1,\ldots,k_p} E_{k_1}(x)E_{k_2}(0)\cdots E_{k_p}(0).$$

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Problem (Inverse Formula)

$$\exists ? P \in \mathbb{R}[x_1,\ldots,x_k] \Rightarrow E_n(x) = P\left(E_{n_1}^{(p_1)}(x),\ldots,E_{n_k}^{(p_k)}(x)\right)?$$

Y-N-Y

Theorem (LJ, V.H.Moll, and C.Vignat, 14')

. $\forall N \in \mathbb{Z}_+$

$$E_n(x) = \frac{1}{N^n} \sum_{\ell=N}^{\infty} p_\ell^{(N)} E_n^{(\ell)} \left(\frac{\ell - N}{2} + Nx \right),$$

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where $$\frac{1}{T_N\left(\frac{1}{z}\right)} = \sum_{\ell=0}^{\infty} p_\ell^{(N)} z^\ell, \quad T_N(\cos\theta) = \cos(N\theta).$$

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$N = 2$:

$$\cos(2\theta) = 2\cos^2\theta - 1 \Rightarrow T_2(z) = 2z^2 - 1.$$

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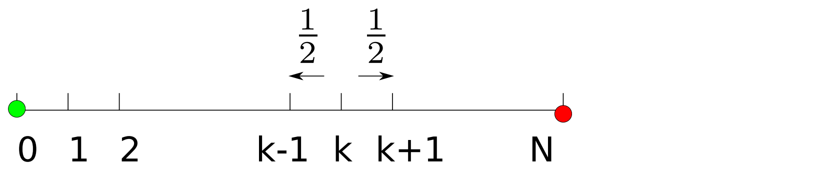
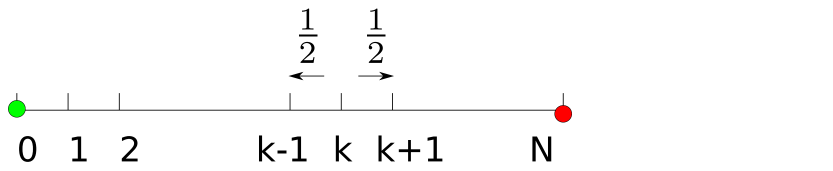
$$\frac{1}{T_2(1/z)} = \frac{z^2}{2 - z^2} = \sum_{k=1}^{\infty} \frac{z^{2k}}{2^k} \Rightarrow p_\ell^{(2)} = \begin{cases} \frac{1}{2^k}, & \text{if } \ell = 2k; \\ 0, & \text{otherwise} \end{cases}$$

Random Walk



Random Walk



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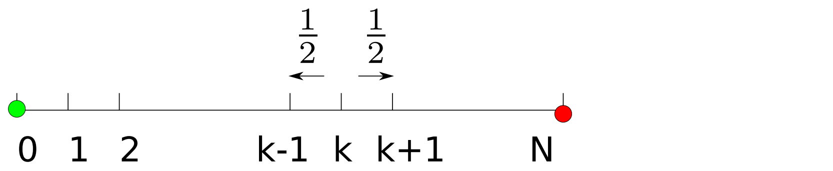

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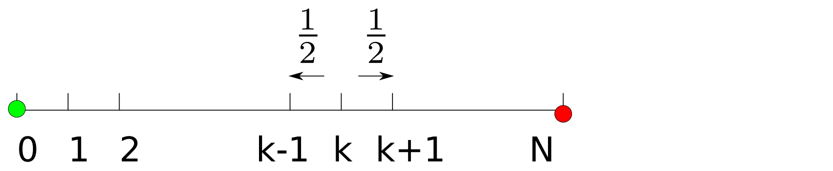

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- let ν_N be the random number of steps for this process.

$$p_\ell^{(N)} = \mathbb{P}(\nu_N = \ell)$$

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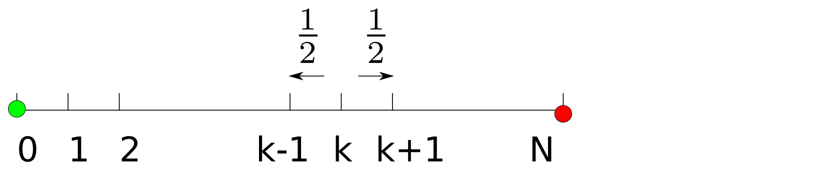

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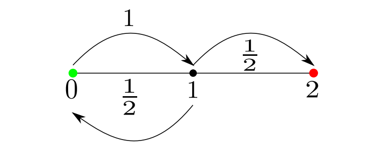

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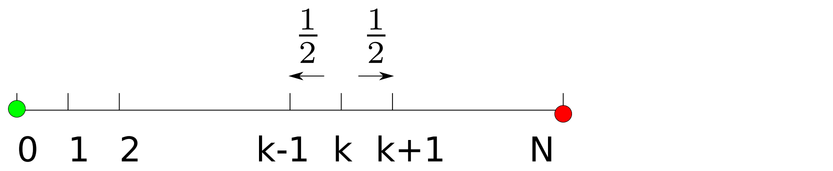

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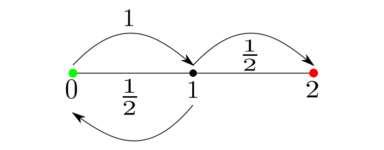

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Probabilistic Interpretation

(1) $L \sim \text{sech}(\pi t)$

[1] L. B. Klebanov, A. V. Kakosyan, S. T. Rachev, and G. Temnov. On a class of distributions stable under random summation. *J. Appl. Prob.*, 49:303–318, 2012.

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(1) $L \sim \operatorname{sech}(\pi t) \Rightarrow E_n(x) = \int_{\mathbb{R}} \left(x + it - \frac{1}{2}\right)^n \operatorname{sech}(\pi t)\, dt.$

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Theorem. [Klebanov et al.[1]]. Let ν_N be an integer valued random variable independent of the L_j 's, defined by the moment generating function:

$$\mathbb{E}\left[z^{\nu_N}\right] = \frac{1}{T_N\left(\frac{1}{z}\right)}.$$

Then, the random variable

$$Z_N = \frac{1}{N}\sum_{j=1}^{\nu_N} L_j$$

has the same hyperbolic secant distribution (as L_j's).

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Proof of the identity

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$$L \sim Z_N = \frac{1}{N} \sum_{j=1}^{\nu_N} L_j$$

$$\Rightarrow \quad x + iL - \frac{1}{2} \sim x + \left(\frac{1}{N} \sum_{j=1}^{\nu_N} iL_j \right) - \frac{1}{2}$$

$$\Rightarrow \quad x + iL - \frac{1}{2} \sim \frac{1}{N} \sum_{j=1}^{\nu_N} \left(iL_j - \frac{\nu_N}{2} + Nx - \frac{N}{2} + \frac{\nu_N}{2} \right)$$

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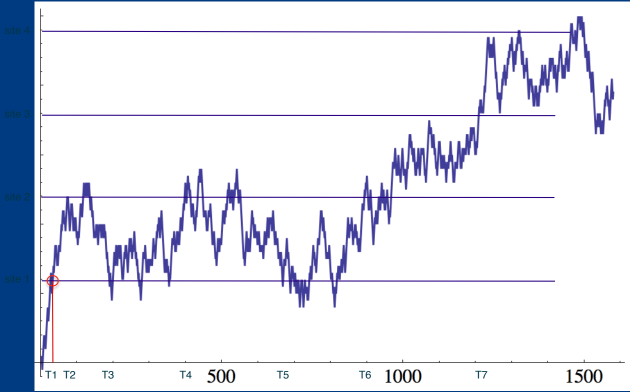
nth moments on both sides.

Reflected Brownian Motion



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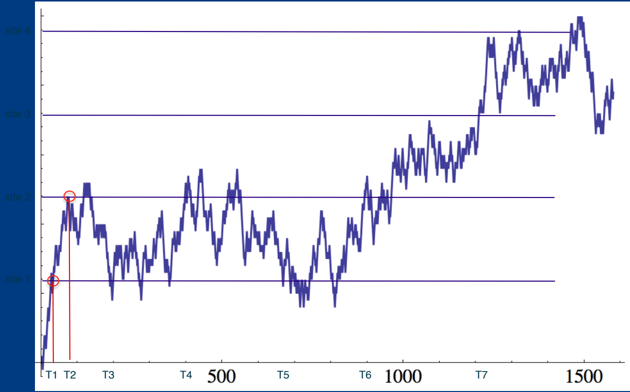

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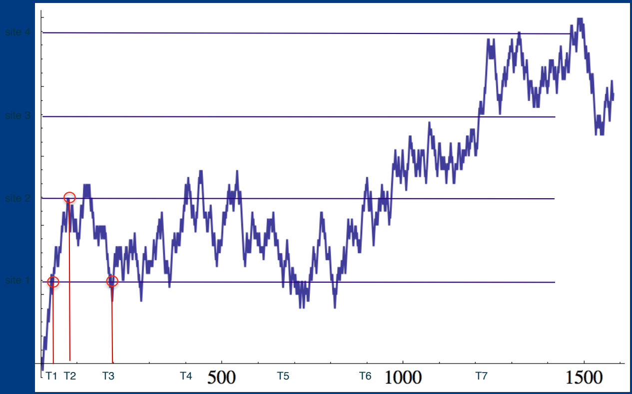

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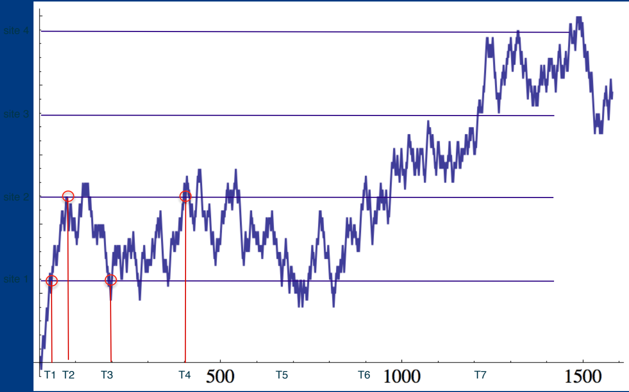

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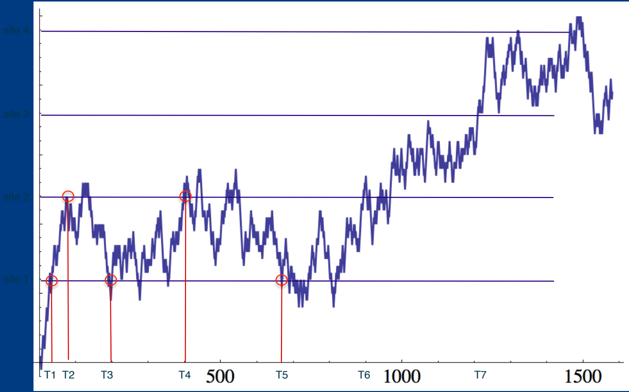

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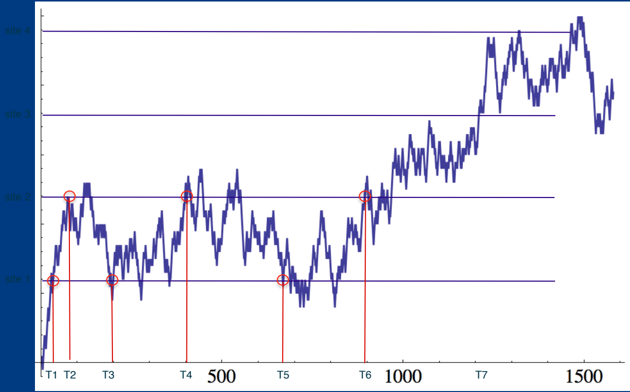

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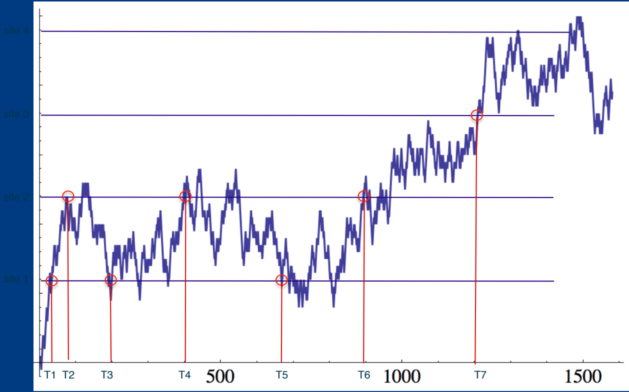

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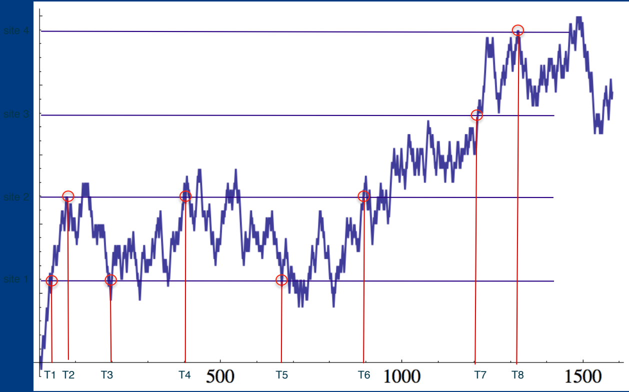
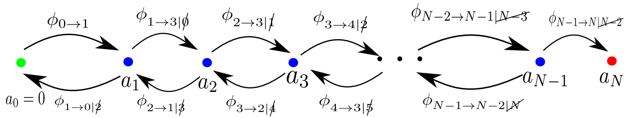
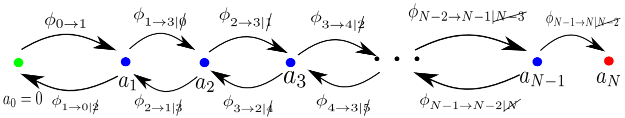



$\phi_{0\to1}$
$\phi_{1\to3|\cancel{0}}$
$\phi_{2\to3|\cancel{1}}$
$\phi_{3\to4|\cancel{2}}$
$\phi_{N-2\to N-1|\cancel{N-3}}$
$\phi_{N-1\to N|\cancel{N-2}}$
a_1
a_2
a_3
a_{N-1}
a_N
$a_0=0$
$\phi_{1\to0|\cancel{2}}$
$\phi_{2\to1|\cancel{3}}$
$\phi_{3\to2|\cancel{4}}$
$\phi_{4\to3|\cancel{5}}$
$\phi_{N-1\to N-2|\cancel{N}}$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$a_0=0$
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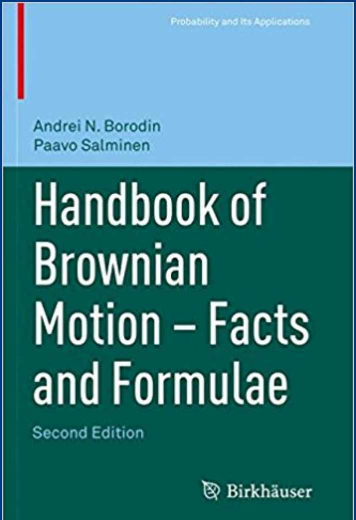
$\phi_{N-1\to N-2|\cancel{N}}$


Probability and Its Applications
Andrei N. Borodin
Paavo Salminen
Handbook of Brownian Motion – Facts and Formulae
Second Edition
Birkhäuser

1-dim, 1-loop

With $p \leq q \leq r$, $w = \sqrt{2\alpha}$

$$\phi_{p\to q} := \mathbb{E}_p\left[e^{-\alpha H_q}\right] = \frac{\cosh(pw)}{\cosh(qw)},$$

$$\phi_{q\to p|\not r} := \mathbb{E}_q\left[e^{-\alpha H_p}|W_t < r\right] = \frac{\sinh((r-q)\,w)}{\sinh((r-p)\,w)},$$

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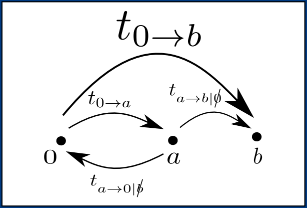

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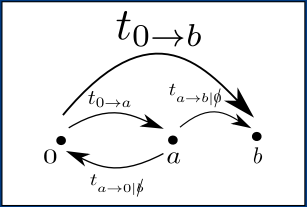

- The hitting time $t_{0\to b}$ can be decomposed as

$$t_{0\to b} = \underbrace{\left(t_{0\to a} + t_{a\to 0|\cancel{b}}\right) + \cdots + \left(t_{0\to a} + t_{a\to 0|\cancel{b}}\right)}_{\ell \text{ copies}} + t_{0\to a} + t_{a\to b|\cancel{0}}$$

1-dim, 1-loop

With $p \leq q \leq r$, $w = \sqrt{2\alpha}$

$$\phi_{p\to q} := \mathbb{E}_p\left[e^{-\alpha H_q}\right] = \frac{\cosh(pw)}{\cosh(qw)},$$

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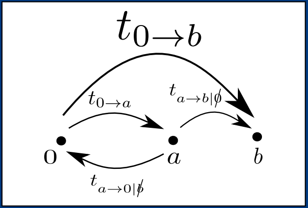

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$$\phi_{0\to b} = \phi_{0\to a}\phi_{a\to b|\not 0}\sum_{\ell=0}^{\infty}\left(\phi_{0\to a}\phi_{a\to 0|\not b}\right)^{\ell}$$

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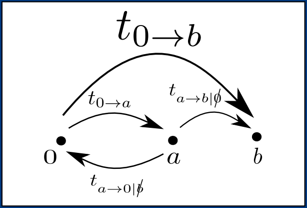

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$$\begin{aligned} \phi_{0\to b} &= \operatorname{sech}(bw), \\ \text{RHS} &= \operatorname{sech}(aw)\cdot\frac{\sinh(aw)}{\sinh(bw)} \sum_{\ell=0}^{\infty}\left[\operatorname{sech}(aw)\cdot\frac{\sinh((b-a)w)}{\sinh(bw)}\right]^{\ell} \end{aligned}$$

1-dim, 1-loop

Proposition (LJ. and C. Vignat, 18'-19')

$$E_n\left(\frac{x}{2b}+\frac{3}{2}-2\frac{a}{b}\right)-E_n\left(\frac{x}{b}+\frac{1}{2}\right)=\frac{(n+1)\left(1-2\,\frac{a}{b}\right)2^n a^n}{b^n}\sum_{\ell=0}^{\infty}\underline{\frac{a}{b}\left(1-\frac{a}{b}\right)^{\ell}}B_n^{(\ell+1)}\left(\frac{x+b}{4a}+\frac{\ell}{2}\right).$$

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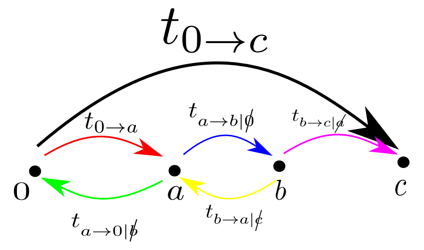
- $\frac{a}{b}\left(1-\frac{a}{b}\right)^{\ell}$ *are the probability weights of a geometric distribution with parameter* a/b.
- *The case* $b=2a$, *i.e., equally distributed sites, gives*

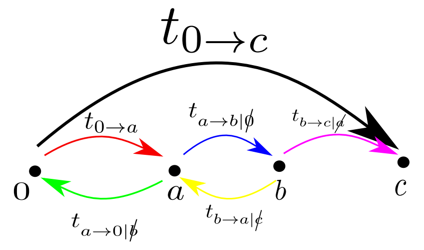
$$0=0.$$

1-dim, 2-loops



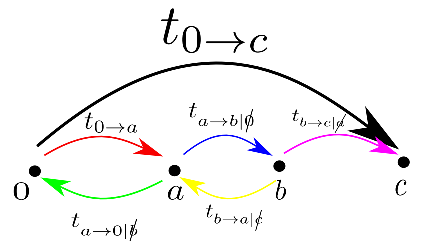
$t \;=\; t$

1-dim, 2-loops



$$t = t + t$$

1-dim, 2-loops



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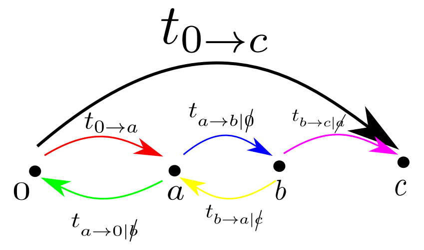

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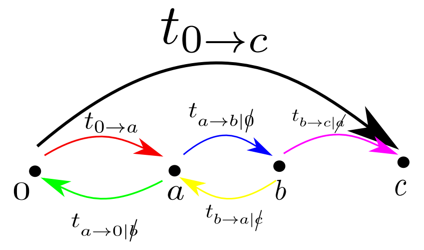

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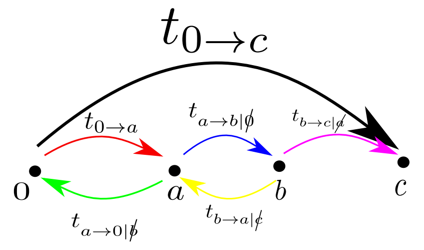

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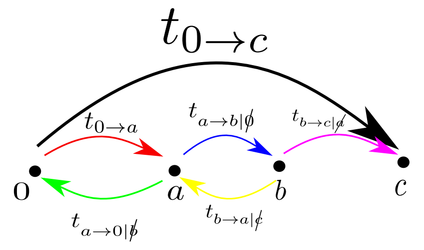

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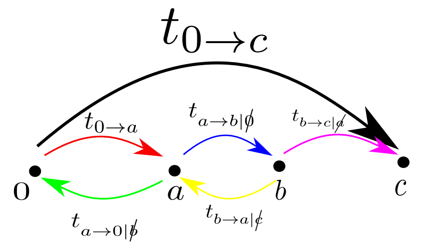
1-dim, 2-loops



$$
\begin{aligned}
t &= t + t + t + t + t + t + \cdots + t \\
&= t + t + t + \underbrace{(t + t) + \cdots + (t + t)}_{k \text{ loops}} + \underbrace{(t + t) + \cdots + (t + t)}_{\ell \text{ loops}}
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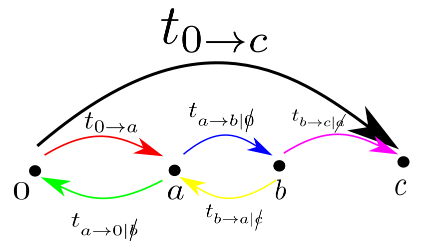
1-dim, 2-loops



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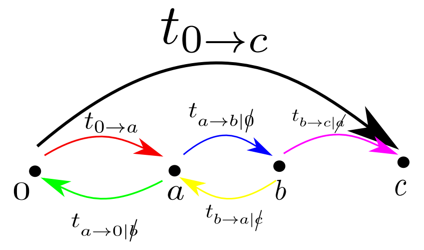
We can generalize it to n-loop model.

1-dim, 2-loops



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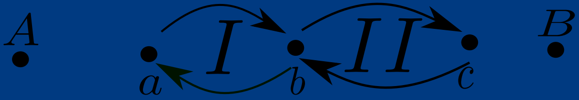
Unfortunately, this is WRONG.....

Two-loops



$$I := \phi_{a\to b|\cancel{A}}\,\phi_{b\to a|\cancel{c}}, \qquad II := \phi_{b\to c|\cancel{a}}\,\phi_{c\to b|\cancel{B}}$$

- k loops of I followed by l loops of II, with $k, l = 0, 1, \ldots,$ which gives

$$\sum_{k.l} I^k II^l = \frac{1}{1-I}\cdot\frac{1}{1-II};$$

- k_1 loops of I followed by l_1 loops of II, then followed by k_2 loops of I and finally followed by l_2 loops of II, with k_1, l_2 nonnegative and k_2, l_1 positive, which gives

$$\sum_{k_1 . l_2 = 0, k_2, l_1 = 1}^{\infty} I^{k_1} II^{l_1} I^{k_2} II^{l_2} = \frac{I\cdot II}{(1-I)^2\,(1-II)^2};$$

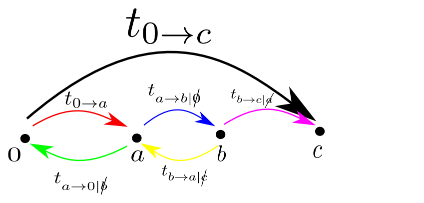
- the general term will be k_1 loops of $I \to l_1$ loops of $II \to \cdots \to k_n$ loops of $I \to l_n$ loops of II, with k_1, l_n nonnegative and the rest indices positive, which gives

$$\frac{(I\cdot II)^{n-1}}{(1-I)^n\,(1-II)^n}.$$

Therefore, loops I and II contribute as

$$\sum_{n=1}^{\infty}\frac{(I\cdot II)^{n-1}}{(1-I)^n\,(1-II)^n} = \frac{1}{1-(I+II)} = \underline{\sum_{k=0}^{\infty}(I+II)^k}.$$

Two Loops



$$\phi = \frac{\phi \cdot \phi \cdot \phi}{1 - \phi\phi - \phi\phi}$$

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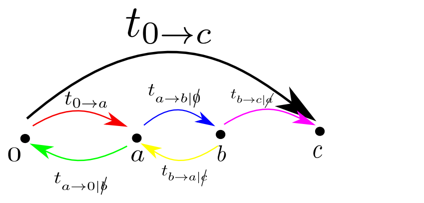

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Proposition (LJ. and C. Vignat, 18'-19')

For any positive integer n,

$$E_n\left(\frac{x}{6}\right) = \sum_{k=0}^{\infty} \frac{3^{k-n}}{4^{k+1}} E_n^{(2k+3)}\left(\frac{x}{2}+k\right).$$

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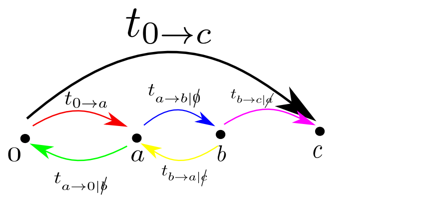

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In general

$$(x + 2c\mathcal{E} + c)^n = \sum_{k=0}^{\infty}\sum_{\ell=0}^{k} q_{k,\ell}\Big[x + 2(b-a)\mathcal{B} + 2(c-b)\mathcal{B}' + a\mathcal{E}^{(\ell)} + 2(b-a)\mathcal{U}^{(l)}$$
$$+ 2a\mathcal{U}'^{(k-\ell)} + 2(b-a)\mathcal{B}'^{(k-\ell)} + q'_{k,\ell}\Big]^n.$$

n loops?

Consider consecutive loops l_1, l_2, $\ldots$, l_n, it seems like the contribution is

$$\sum_{k=0}^{\infty} \left(\sum_{\ell=1}^{n} l_\ell \right)^k = \frac{1}{1 - (l_1 + \cdots + l_n)}. \qquad (*)$$

- It feels right.

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- In general sites $0, 1, \ldots, N$:

$$\frac{1}{\cosh(Nw)} \overset{??}{=} \frac{\frac{1}{\cosh w}\cdot\left(\frac{\sinh w}{\sinh(2w)}\right)^N}{1-\left(\frac{1}{\cosh(w)}\frac{\sinh(w)}{\sinh(2w)}+(N-1)\frac{\sinh(w)}{\sinh(2w)}\frac{\sinh(w)}{\sinh(2w)}\right)}$$

$$= \frac{\frac{1}{2^N\cosh^{N+1} w}}{1-\frac{N+3}{4}\cosh^N w}.$$

This shows $(*)$ is not correct.

$$\frac{1}{\cosh(Nw)} = \frac{1}{\cos(Niw)} = \frac{1}{T_N(\cos(iw))} = \frac{1}{T_N(\cosh w)}.$$

n-loop case

Dr. Italo Simonelli

Heng Yue

Theorem (LJ, I.Simonelli, and H.Yue, 21'+)

$$\phi_{0\to a_n} = \phi_{0\to a_1}\phi_{a_1\to a_2|\cancel{\emptyset}}\cdots\phi_{a_{n-1}\to a_n|\cancel{a_{n-2}}}\cdot\frac{1}{1-P(L_1,\ldots,L_n)},$$

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for the condition $*$ *given by (1)* $\ell = 1,2,\ldots,n$; *(2) and* $j_1 < j_2 - 1$, $j_2 < j_3 - 1$,. . . ,$j_{\ell-1} < j_\ell - 1$.

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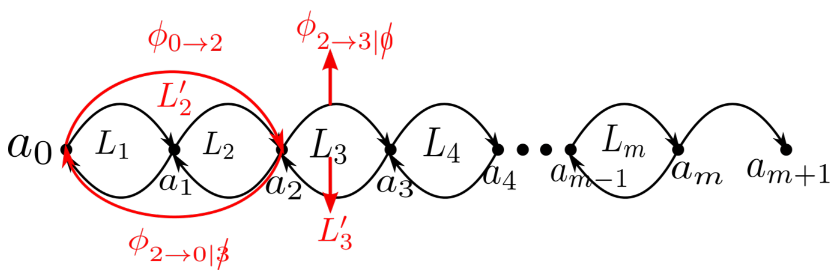

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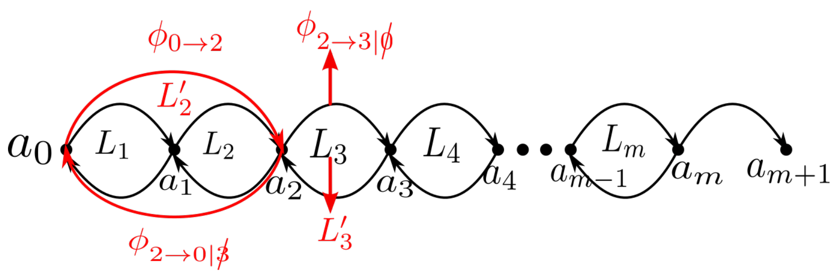

1. Induction.

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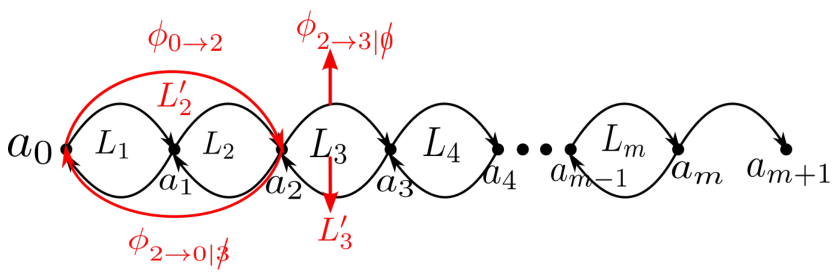

1. Induction. The tricky part is, if we "glue" the first two loops together; or ignore site a_1, $\phi_{2\to 3|\not{1}}$ should be replaced by $\phi_{2\to 3|\not{0}}$.

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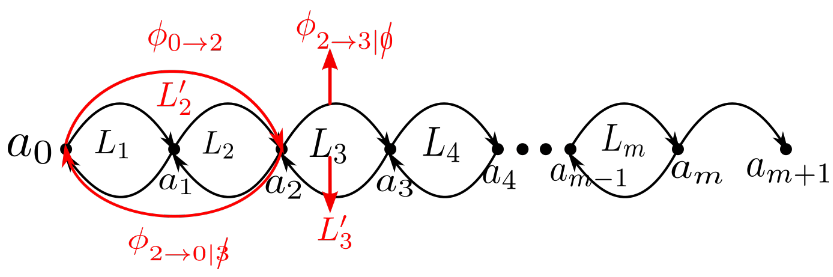

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2. Inclusion-exclusion principle

Theorem (LJ, I.Simonelli, and H.Yue, 21'+)

Let $M = \lceil m \rceil - 1$, and M' be the largest odd number less or equal to M, then

$$
\begin{aligned}
E_n\left(\frac{x}{m+1}\right) = {} & \frac{1}{(m+1)^n} \sum_{k=0}^{\infty} \frac{(m+1)^k}{2^{2k+m}} \sum_{n_1,\dots,n_M=0}^{k} \binom{k}{n_1,\dots,n_M} \\
& \times (-1)^{n_1+n_3+\cdots+n_{M'}} \frac{4^{n_1+\cdots+n_M}}{(m+1)^{n_1+\cdots+n_M}} \\
& \times \left(\frac{\binom{m-2}{1}}{2^3} + \frac{\binom{m-2}{2}}{2^4} \right)^{n_1} \cdots \left(\frac{\binom{m-M-1}{M}}{2^{2M+1}} + \frac{\binom{m-M-1}{M+1}}{2^{2M+2}} \right)^{n_M} \\
& \times E_n^{(2k+2n_1+4n_2+\cdots+2Mn_M+m)}\left(k + n_1 + 2n_2 + \cdots + Mn_M + x\right).
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$$\times(-1)^{n_1+n_3+\cdots+n_{M'}}\frac{4^{n_1+\cdots+n_M}}{(m+1)^{n_1+\cdots+n_M}}$$
$$\times\left(\frac{\binom{m-2}{1}}{2^3}+\frac{\binom{m-2}{2}}{2^4}\right)^{n_1}\cdots\left(\frac{\binom{m-M-1}{M}}{2^{2M+1}}+\frac{\binom{m-M-1}{M+1}}{2^{2M+2}}\right)^{n_M}$$
$$\times E_n^{(2k+2n_1+4n_2+\cdots+2Mn_M+m)}\left(k+n_1+2n_2+\cdots+Mn_M+x\right).$$

$$E_n(x) = \frac{1}{4^n}\sum_{k=0}^{\infty}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\frac{2^n}{8^{\ell+1}}E_n^{(2k+2\ell)}\left(4x+k+\ell\right).$$

$$E_n\left(x\right) = \sum_{k=0}^{\infty}\frac{5^{k-n}}{4^{k+\ell+2}}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}E_n^{(2\ell+2k+5)}\left(5x+\ell+k\right).$$

Generalization

- Bessel process in $\mathbb{R}^n$:

$$R_t^{(n)} := \sqrt{\left(W_t^{(1)}\right)^2 + \cdots + \left(W_t^{(n)}\right)^2}$$

- Moment generating functions for hitting times: $H_z := \min_s \left\{ R_s^{(n)} = z \right\}$.

$$\mathbb{E}_x \left(e^{-\alpha H_z}; \sup_{0 \le s \le H_z} R_s^{(n)} < y \right) = \begin{cases} \frac{x^{-\nu} I_\nu(xw)}{z^{-\nu} I_\nu(zw)}, & 0 \le x \le z \le y; \\ \frac{S_\nu(yw,xw)}{S_\nu(yw,zw)}, & z \le x \le y, \end{cases}$$

- $n = 2 + 2\nu$ for $\nu \ge 0$

$$S_\nu(x, y) := (xy)^{-\nu} \left[I_\nu(x) K_\nu(y) - K_\nu(x) I_\nu(y) \right],$$

and

$$I_\nu(x) = \sum_{\ell=0}^{\infty} \frac{1}{\ell! \Gamma(\ell + \nu + 1)} \left(\frac{x}{2} \right)^{2\ell+\nu}, \quad K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\nu\pi)}.$$

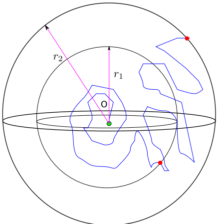
$n = 3 \Leftrightarrow \nu = 1/2$

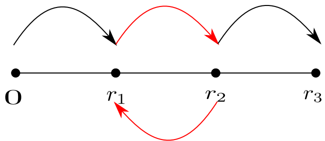
- $$I_{\frac{1}{2}}(x) = \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2m+\frac{1}{2}}}{m!\Gamma\left(m+\frac{3}{2}\right)} = \sqrt{\frac{2}{x\pi}} \sinh(x)$$

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t - 1}e^{tx} = \frac{te^{tx}e^{-\frac{t}{2}}}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} = \frac{te^{t\left(x-\frac{1}{2}\right)}}{2}\sinh\left(\frac{t}{2}\right)$$

$$K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}$$

$$\mathbb{E}_x\left(e^{-\alpha H_z}; \sup_{0\le s\le H_z} R_s^{(3)} < y\right) = \begin{cases} \frac{z\sinh(xw)}{x\sinh(zw)}, & 0 \le x \le z \le y \\ \frac{z\sinh((y-x)w)}{x\sinh((y-z)w)}, & z \le x \le y \end{cases}$$





Proposition (LJ and C. Vignat, 19')

$$\frac{3^{n+1}}{n+1}\left[B_{n+1}\left(\frac{x}{6}+\frac{5}{6}\right)-B_{n+1}\left(\frac{x}{6}+\frac{1}{2}\right)\right]=\sum_{k\geq 0}\frac{3}{4}\left(\frac{1}{4}\right)^{k}E_{n}^{(2k+2)}\left(\frac{x+3+2k}{2}\right)$$

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Theorem (LJ, I.Simonelli, and H.Yue, 21'+)

For the 3-dimensional Bessel process on sites $0,1,\ldots,m+2$*, we have*

$$\phi_{0\to(m+2)}=\phi_{0\to 1}\prod_{j=1}^{m+1}\phi_{j\to(j+1)|j-1}\frac{1}{1-\sum_{k=1}^{m}\sum_{(k,l,n)}}.$$

Theorem (LJ, I.Simonelli, and H.Yue, 21'+)

Let $M = \lceil m \rceil - 1$, *and* M' *be the largest odd number less or equal to* M, *then*

$$\begin{aligned} & B_{n+1}\left(\frac{2+x}{m+2}\right) - B_{n+1}\left(\frac{x}{m+2}\right) \\ &= \frac{n+1}{(m+2)^n} \sum_{k=0}^{\infty} \frac{m^k}{2^{2k+m}} \sum_{n_1,\ldots,n_M=0}^{k} \binom{k}{n_1,\ldots,n_M} (-1)^{n_1+n_3+\cdots+n_{M'}} \\ &\quad \times \frac{\binom{m-1}{2}^{n_1}\binom{m-2}{3}^{n_2}\cdots\binom{m-M}{M}^{n_M}}{4^{n_1+2n_2+\cdots+Mn_M}\, m^{n_1+\cdots+n_M}} \\ &\quad \times E_n^{(2k+2n_1+4n_2+\cdots+2Mn_M+m)}(k+n_1+2n_2+\cdots+Mn_M+x). \end{aligned}$$

Theorem (LJ, I.Simonelli, and H.Yue, 21'+)

Let $M = \lceil m \rceil - 1$, and M' be the largest odd number less or equal to M, then

$$
\begin{aligned}
& B_{n+1}\left(\frac{2+x}{m+2}\right) - B_{n+1}\left(\frac{x}{m+2}\right) \\
&= \frac{n+1}{(m+2)^n} \sum_{k=0}^{\infty} \frac{m^k}{2^{2k+m}} \sum_{n_1,\dots,n_M=0}^{k} \binom{k}{n_1,\dots,n_M} (-1)^{n_1+n_3+\cdots+n_{M'}} \\
&\quad \times \frac{\binom{m-1}{2}^{n_1}\binom{m-2}{3}^{n_2}\cdots\binom{m-M}{M}^{n_M}}{4^{n_1+2n_2+\cdots+Mn_M}\, m^{n_1+\cdots+n_M}} \\
&\quad \times E_n^{(2k+2n_1+4n_2+\cdots+2Mn_M+m)}(k+n_1+2n_2+\cdots+Mn_M+x).
\end{aligned}
$$

$$
B_{n+1}\left(\frac{x+2}{5}\right) - B_{n+1}\left(\frac{x}{5}\right) = \frac{n+1}{5^n}\sum_{k=0}^{\infty}\frac{3^k}{2^{2k+3}}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{\ell}\frac{1}{12^{\ell}}E_n^{(2k+2\ell+3)}(k+\ell+x).
$$

Thank you for your attention