Continued Fractions I

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DKU Discrete Math Seminar

March 11th, 2022

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Rogers–Ramanujan continued fraction

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$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} = 1 + q + q^2 + q^3 + 2q^4 + \cdots$$

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty} = 1 + q^2 + q^3 + q^4 + \cdots$$

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Some Results

Theorem

Let

$$\frac{A_n}{B_n} = q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \cfrac{1}{\ddots_{\, q_4 + \cfrac{1}{q_{n-1} + \frac{1}{q_n}}}}}}.$$

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- *For* $n \geq 3$, $A_n = A_{n-1}q_n + A_{n-2}$ *and* $B_n = B_{n-1}q_n + B_{n-2}$.

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Theorem

Assume $\alpha = \lim_{n \to \infty} \frac{A_n}{B_n}$ *exists. Then,*

$$\frac{1}{B_n(B_n + B_{n+1})} < \left| \alpha - \frac{A_n}{B_n} \right| < \frac{1}{B_n B_{n+1}}.$$

Functions

Let

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1} \quad \text{and} \quad N_n(x) = \sum_{k=0}^{n} N(n,k)x^k.$$

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$$\mathcal{N}(x,t) := \sum_{n=0}^{\infty} N_n(x)t^n = \cfrac{1}{1-\cfrac{xt}{1-\cfrac{t}{1-\cfrac{xt}{1-\cfrac{t}{1-\ddots}}}}}$$

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Definition

$N(n,k)$ are called Narayana numbers and $N_n(x)$ are Narayana polynomials.

Narayana numbers

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- Dyck paths of length $2n$ with k peaks
- Non-crossing partitions of $[n]$ with k blocks
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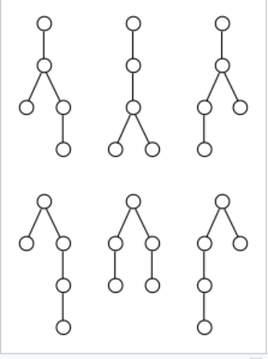

from Wikipedia

Dyck path

Definition

A Dyck path of length $2n$ is a path in the right quadrant $\mathbb{N}^2$ from $(0,0)$ to $(2n,0)$ using steps $(1,1)$ "rise" and $(1,-1)$ "fall"

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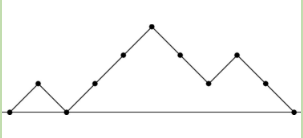

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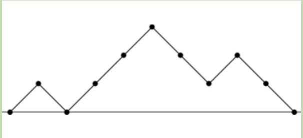


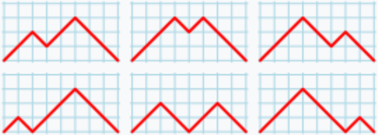
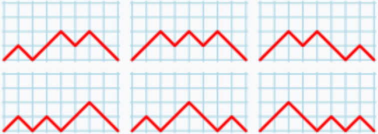
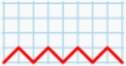
Theorem (Flajolet 1980)

As an identity in $\mathbb{Z}[\vec{\alpha}][[t]]$, we have

$$\frac{1}{1-\frac{\alpha_1 t}{1-\frac{\alpha_2 t}{1-\ddots}}} = \sum_{n=0}^{\infty} S_n(\alpha_1,\ldots,\alpha_n)t^n,$$

where $S_n(\alpha_1,\ldots,\alpha_n)$ is the generating polynomial for Dyck paths of length $2n$ in which each fall starting at height i gets weight α_i.

$n = 4$

$\mathrm{N}(4, k)$	**Paths**
$N(4, 1)$ = **1** path with 1 peak	
$N(4, 2)$ = **6** paths with 2 peaks:	
$N(4, 3)$ = **6** paths with 3 peaks:	
$N(4, 4)$ = **1** path with 4 peaks:	

Wikipedia

Proof

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$$\mathcal{N}(x,t) = 1 + xt\mathcal{N}(x,t)^2 - xt\mathcal{N}(x,t) + t\mathcal{N}(x,t).$$

Elementary "renewal" argument on Dyck paths implies

$$\mathcal{N} = \frac{1}{1 - tx - t(\mathcal{N} - 1)},$$

Proof

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Elementary "renewal" argument on Dyck paths implies

$$\mathcal{N} = \frac{1}{1 - tx - t(\mathcal{N} - 1)},$$

which means

$$\mathcal{N} = \frac{1}{1 - \frac{xt}{1 - t\mathcal{N}}}$$

$$(m_n)_{n=0}^{\infty} \sim m_n = \int_{\mathbb{R}} x^n \mathrm{d}\mu(x)$$

$$(m_n)_{n=0}^{\infty} \sim m_n = \int_{\mathbb{R}} x^n \mathrm{d}\mu(x) \quad \stackrel{?}{\Rightarrow} \quad (P_n(x))_{n=1}^{\infty} \sim \int_{\mathbb{R}} P_n(x) P_m(x) \,\mathrm{d}\mu(x) = C_n \delta_{m,n}$$

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&\Rightarrow \quad P_{n+1}(x) = (x + s_n)\, P_n(x) + t_n P_{n-1}(x)
\end{aligned}
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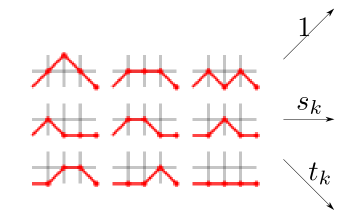
$$\Rightarrow \quad \boxed{\sum_{n=0}^{\infty} m_n x^n = \cfrac{m_0}{1 + s_0 x + \cfrac{t_1 x^2}{1 + s_1 x + \cfrac{t_2 x^2}{1 + \cdots}}}}$$

$$M_{n+1,k} = M_{n,k-1} + s_k M_{n,k} + t_{k+1} M_{n,k+1}$$



$$\boxed{\sum_{n=0}^{\infty} M_{n,0} z^n = \cfrac{1}{1 - s_0 z - \cfrac{t_1 z^2}{1 - s_1 z - \cfrac{t_2 z^2}{\cdots}}}}$$

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- Stieltjes type (S-fractions):

$$f(t) = \cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cfrac{\alpha_3 t}{1 - \cdots}}}}$$

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- Thron type (T-fractions):

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Contractions

- even contraction

$$\frac{1}{1-\frac{\alpha_1 t}{1-\frac{\alpha_2 t}{1-\frac{\alpha_3 t}{1-\cdots}}}} = \frac{1}{1-\alpha_1 t-\frac{\alpha_1\alpha_2 t^2}{1-(\alpha_2+\alpha_3)t-\frac{\alpha_3\alpha_4 t^2}{1-(\alpha_4+\alpha_5)t-\frac{\alpha_5\alpha_6 t^2}{1-\cdots}}}}$$

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- odd contraction

$$\frac{1}{1-\frac{\alpha_1 t}{1-\frac{\alpha_2 t}{1-\frac{\alpha_3 t}{1-\cdots}}}} = 1+\frac{\alpha_1 t}{1-(\alpha_1+\alpha_2)\,t-\frac{\alpha_2\alpha_3 t^2}{1-(\alpha_3+\alpha_4)t-\frac{\alpha_4\alpha_5 t^2}{1-\cdots}}}$$

Hankel determinants

Given a sequence $\mathbf{a} = (a_0, a_1, \ldots)$, the nth Hankel determinant of $\mathbf{a}$ is defined by

$$H_n(\mathbf{a}) = H_n\left(a_k\right) := \det_{0 \le i,j \le n} \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{pmatrix}$$

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For Catalan number $C_n = \binom{2n}{n}/(n+1)$, $H_n(C_k) =$

Orthogonal polynomials

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$$P_n(y) = \frac{1}{H_{n-1}(\mathbf{a})} \det_{0 \leq i,j \leq n} \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-1} \\ 1 & y & \cdots & y^n \end{pmatrix}.$$

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$$\sum_{n=0}^{\infty} a_n x^n = \cfrac{a_0}{1 + s_0 x + \cfrac{t_1 x^2}{1 + s_1 x + \cfrac{t_2 x^2}{1 + \cdots}}}$$