Continued Fractions, II

Lin Jiu

March 25th, 2022

Hankel Determinants

Given a sequence $\mathbf{a} = (a_0, a_1, \ldots)$, the nth Hankel determinant of $\mathbf{a}$ is defined by

$$H_n(\mathbf{a}) = H_n\left(a_k\right) := \det_{0 \le i,j \le n} \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{pmatrix}$$

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A Dyck path of length $2n$ is a path in the right quadrant $\mathbb{N}^2$ from $(0,0)$ to $(2n,0)$ using steps $(1,1)$ "rise" and $(1,-1)$ "fall"

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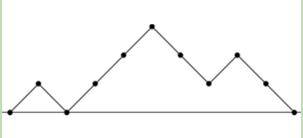

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Hence

$$C(t) = \frac{1 - \sqrt{1-4t}}{2t}.$$

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Theorem (Flajolet 1980)

As an identity in $\mathbb{Z}[\vec{\alpha}][[t]]$, we have

$$\cfrac{1}{1-\cfrac{\alpha_1 t}{1-\cfrac{\alpha_2 t}{1-\ddots}}} = \sum_{n=0}^{\infty} S_n(\alpha_1, \ldots, \alpha_n) t^n,$$

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\text{[even contraction]} &= \frac{1}{1-t-\frac{t^2}{1-2t-\frac{t^2}{1-2t-\frac{t^2}{1-\ddots}}}}
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Hankel Determinants Again

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Given a sequence $\mathbf{a} = (a_k)_{k=0}^{\infty}$ *with its monic orthogonal polynomials* $P_n(y)$*,*

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Given a sequence $\mathbf{a} = (a_k)_{k=0}^{\infty}$ *with its monic orthogonal polynomials* $P_n(y)$*, namely,*

$$P_n(y) = \frac{1}{H_{n-1}(\mathbf{a})} \det_{0 \le i,j \le n} \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-1} \\ 1 & y & \cdots & y^n \end{pmatrix},$$

satisfying the three-term recurrence

$$P_{n+1}(y) = (y + s_n)P_n(y) - t_n P_{n-1}(y),$$

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$$H_n(\mathbf{a}) = c_0^{n+1} t_1^n t_2^{n-1} \cdots t_{n-1}^2 t_n.$$

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Theorem (K. Dilcher and L. J)

Let $b_k = B_{2k+1}\left(\frac{x+1}{2}\right)$, *then*

$$H_n\left(b_k\right) = (-1)^{\binom{n+1}{2}}\left(\frac{x}{2}\right)^{n+1}\prod_{\ell=1}^{n}\left(\frac{\ell^4\left(x^2-\ell^2\right)}{4(2\ell+1)(2\ell-1)}\right)^{n+1-\ell}.$$

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Moreover, the orthogonal polynomial w. r. t. b_k*satisfies the recurrence*

$$P_{n+1}(y) = (y-\sigma_n)P_n(y) - \tau_n P_{n-1}(y),$$

where

$$\sigma_n = \binom{n+1}{2} - \frac{x^2-1}{4} \quad \textit{and} \quad \tau_n = \frac{n^4\left(x^2-n^2\right)}{4(2n+1)(2n-1)}$$

(Not the Complete) Proof

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$$\log \Gamma(z+x) = \left(z + x - \frac{1}{2}\right)\log z - z + \frac{\log(2\pi)}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} B_{n+1}(x)}{n(n+1)z^n}$$

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Then,

$$\sum_{k=0}^{\infty} B_{2k+1}\left(\frac{x+1}{2}\right)z^{2k} = \frac{1}{2z^2}\left(\psi'\left(\frac{1}{z}+\frac{1-x}{2}\right)-\psi'\left(\frac{1}{z}+\frac{1+x}{2}\right)\right),$$

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Finally,

$$\sum_{k=0}^{\infty} B_{2k+1}\left(\frac{x+1}{2}\right) z^{2k} = \cfrac{\frac{x}{2}}{1+\sigma_0 z^2 + \cfrac{\tau_1 z^4}{1+\sigma_1 z^2 + \cfrac{\tau_2 z^4}{1+\sigma_2 z^2 + \ddots}}}.$$

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$$\sum_{k=0}^{\infty} \frac{1}{(s-b+2k+1)^2} - \sum_{k=0}^{\infty} \frac{1}{(s+b+2k+1)^2}$$

$$= \cfrac{b}{1\left(s^2-b^2+1\right) - \cfrac{4(1^2-b^2)1^4}{4(s^2-b^2+5) - \cfrac{4(2^2-b^2)2^4}{5(s^2-b^2+13) - \ddots}}}.$$

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- Open Question 1.5: Given a function $f(z)$, and a continued fraction expression, how can we show they are equal? In what sense?

Theorem (K. Dilcher and L. J)

$$H_n\left(B_{2k+1}\left(\frac{x+1}{2}\right)\right) = (-1)^{\binom{n+1}{2}}\left(\frac{x}{2}\right)^{n+1}\prod_{\ell=1}^{n}\left(\frac{\ell^4\left(x^2-\ell^2\right)}{4(2\ell+1)(2\ell-1)}\right)^{n+1-\ell}.$$

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Theorem (L. J and April Li)

$$H_n\left(\frac{B_{2k+1}\left(\frac{x+1}{2}\right)}{2k+1}\right) = \left(\frac{x}{2}\right)^{n+1}\prod_{\ell=1}^{n}\left(\frac{(2\ell)^2(2\ell-1)^2\left(x^2-(2\ell-1)^2\right)\left(x^2-(2\ell)^2\right)}{16(4\ell-3)(4\ell-1)^2(4\ell+1)}\right)^{n+1-\ell}.$$

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- Open Question 2: What is the relation between

$$H_n(a_k) \quad \text{and} \quad H_n\left(\frac{a_k}{k}\right)?$$

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Definition

The Euler polynomials $E_n(x)$ and Euler numbers E_n are defined by

$$\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \quad \text{and} \quad \frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

$$H_n(a_k) = (-1)^{\varepsilon(n)} a^{n+1} \prod_{\ell=1}^{n} b(\ell)^{n+1-\ell}$$

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E_k	$\binom{n+1}{2}$	1	ℓ^2
$E_k(x)$	$\binom{n+1}{2}$	1	$\frac{\ell^2}{4}$
$E_{k+1}(1)$	$\binom{n+1}{2}$	$\frac{1}{2}$	$\frac{\ell(\ell+1)}{4}$
E_{2k}	0	1	$(2\ell-1)^2(2\ell)^2$
$E_{2k+1}(1)$	0	$\frac{1}{2}$	$\frac{\ell^2(2\ell-1)(2\ell+1)}{4}$
E_{2k+2}	$n+1$	1	$(2\ell)^2(2\ell+1)^2$
$E_{2k+3}(1)$	$n+1$	$\frac{1}{4}$	$\frac{\ell(\ell+1)(2\ell+1)^2}{4}$
$(2k+1)E_{2k}$	0	1	$(2\ell)^4$
$(2k+2)E_{2k+1}(1)$	0	1	$\ell^3(\ell+1)$

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- $$H_n(a_k) \Rightarrow H_n(\text{"}a_{k-1}\text{"}).$$

Contractions

- even contraction

$$\frac{1}{1-\frac{\alpha_1 t}{1-\frac{\alpha_2 t}{1-\frac{\alpha_3 t}{1-\cdots}}}} = \frac{1}{1-\alpha_1 t-\frac{\alpha_1\alpha_2 t^2}{1-(\alpha_2+\alpha_3)t-\frac{\alpha_3\alpha_4 t^2}{1-(\alpha_4+\alpha_5)t-\frac{\alpha_5\alpha_6 t^2}{1-\cdots}}}}$$

- odd contraction

$$\frac{1}{1-\frac{\alpha_1 t}{1-\frac{\alpha_2 t}{1-\frac{\alpha_3 t}{1-\cdots}}}} = 1+\frac{\alpha_1 t}{1-(\alpha_1+\alpha_2)\,t-\frac{\alpha_2\alpha_3 t^2}{1-(\alpha_3+\alpha_4)t-\frac{\alpha_4\alpha_5 t^2}{1-\cdots}}}$$

Left-shifted

$$\sum_{n=0}^{\infty} a_n x^n = \cfrac{1}{1 + s_0 x + \cfrac{t_1 x^2}{1+s_1 x+\frac{t_2 x^2}{1+\cdots}}}$$

Left-shifted

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1}{1 + s_0 x + \frac{t_1 x^2}{1 + s_1 x + \frac{t_2 x^2}{1 + \cdots}}} \qquad \text{assume } a_0 = 1.$$

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Then,

$$\sum_{n=1}^{\infty} a_n x^{n-1}$$

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Then,

$$\sum_{n=1}^{\infty} a_n x^{n-1} = \frac{\cfrac{1}{1 + s_0 x + \cfrac{t_1 x^2}{1 + s_1 x + \cfrac{t_2 x^2}{1 + \cdots}}} - 1}{x}$$

Open Question(s)

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-

	$p = 1$	$p = 2$	$p = 3$	$p = 4$	$p = 5$
$n = 1$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{5}{12}$
$n = 2$	$\frac{4}{15}$	$\frac{13}{30}$	$\frac{3}{5}$	$\frac{23}{30}$	$\frac{14}{15}$
$n = 3$	$\frac{81}{140}$	$\frac{372}{455}$	$\frac{1339}{1260}$	$\frac{2109}{1610}$	$\frac{1527}{980}$
$n = 4$	$\frac{64}{63}$	$\frac{3736}{2821}$	$\frac{138688}{84357}$	$\frac{668543}{339549}$	$\frac{171830}{74823}$
$n = 5$	$\frac{625}{396}$	$\frac{1245075}{636988}$	$\frac{299594775}{127670972}$	$\frac{42601023200}{15509529057}$	$\frac{3638564965}{1154491404}$

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	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$
$n=1$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{5}{12}$
$n=2$	$\frac{4}{15}$	$\frac{13}{30}$	$\frac{3}{5}$	$\frac{23}{30}$	$\frac{14}{15}$
$n=3$	$\frac{81}{140}$	$\frac{372}{455}$	$\frac{1339}{1260}$	$\frac{2109}{1610}$	$\frac{1527}{980}$
$n=4$	$\frac{64}{63}$	$\frac{3736}{2821}$	$\frac{138688}{84357}$	$\frac{668543}{339549}$	$\frac{171830}{74823}$
$n=5$	$\frac{625}{396}$	$\frac{1245075}{636988}$	$\frac{299594775}{127670972}$	$\frac{42601023200}{15509529057}$	$\frac{3638564965}{1154491404}$

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n	$d_n^{(1)}$
0	$\frac{1}{4}(x^2-3)$
1	$\frac{1}{16}(x^4-18x^2+41)$
2	$\frac{1}{64}(x^6-53x^4+655x^2-1323)$
3	$\frac{1}{256}(x^8-116x^6+3958x^4-41364x^2+77841)$

Open Question(s)

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	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$
$n=1$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{5}{12}$
$n=2$	$\frac{4}{15}$	$\frac{13}{30}$	$\frac{3}{5}$	$\frac{23}{30}$	$\frac{14}{15}$
$n=3$	$\frac{81}{140}$	$\frac{372}{455}$	$\frac{1339}{1260}$	$\frac{2109}{1610}$	$\frac{1527}{980}$
$n=4$	$\frac{64}{63}$	$\frac{3736}{2821}$	$\frac{138688}{84357}$	$\frac{668543}{339549}$	$\frac{171830}{74823}$
$n=5$	$\frac{625}{396}$	$\frac{1245075}{636988}$	$\frac{299594775}{127670972}$	$\frac{42601023200}{15509529057}$	$\frac{3638564965}{1154491404}$

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- “Weakly increasing trees on a multiset” by Z. Lin, J. Ma, S-M. Ma, and Y. Zhou.