Hankel Determinants of Certain Sequences of Bernoulli and Euler Polynomials

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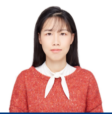
Acknowledgments

Dr. Karl Dilcher, Dalhousie University



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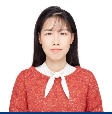

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Hankel Determinants

Definition

A *Hankel matrix* or *persymmetric matrix* is a symmetric matrix which has constant entries along its antidiagonals; in other words, it is of the form

$$(c_{i+j})_{0\le i,j\le n} = \begin{pmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n+1} & \cdots & c_{2n} \end{pmatrix}.$$

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The Hankel determinant of a given sequence $(c_k)_{k\ge 0}$ is the determinant of the Hankel matrix.

$$H_n(c_k) = \det(c_{i+j})_{0\le i,j\le n}.$$

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$$c_0 = 1, c_1 = 1, c_2 = 2, c_3 = 5, c_4 = 14, \ldots$$

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$$\det\begin{pmatrix} 2 & 5 \\ 5 & 14 \end{pmatrix} - \det\begin{pmatrix} 1 & 2 \\ 5 & 14 \end{pmatrix} + 2\det\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$$

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  $\det\begin{pmatrix} 2 & 5 \\ 5 & 14 \end{pmatrix} - \det\begin{pmatrix} 1 & 2 \\ 5 & 14 \end{pmatrix} + 2\det\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = 3 - 4 + 2 \cdot 1 = 1.$

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Theorem

$H_n(C_k) = 1$, *for any* $n = 0, 1, 2, \ldots.$

Bernoulli Polynomials

Definition

The Bernoulli polynomials are defined by the exponential generating functions

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{e^{xt}t}{e^t - 1}$$

and the Bernoulli numbers are $B_n = B_n(0)$.

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Theorem

$$H_n(B_k(x)) = H_n(B_k) = (-1)^{\binom{n+1}{2}} \prod_{\ell=1}^{n} \left(\frac{\ell^4}{4(2\ell+1)(2\ell-1)} \right)^{n+1-\ell}.$$

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Remark

$$B_{12} = -\frac{691}{2\cdot 3\cdot 5\cdot 7\cdot 13} \quad \text{and} \quad H_{10}(B_k) = -\frac{2^{42}\cdot 3^{15}\cdot 5^4}{11^{11}\cdot 13^9\cdot 17^5\cdot 19^3}.$$

Theorem

For any sequence c_n,

$$c_n(x) = \sum_{k=0}^{n} \binom{n}{k} c_k x^{n-k} \Rightarrow H_n(c_k(x)) = H_n(c_k).$$

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Fact

n	$H_n\left(B_{2k+1}\left(\frac{x+1}{2}\right)\right)$
0	$\frac{x}{2}$
1	$-\frac{1}{48}x^2(x^2-1)$
2	$-\frac{1}{4320}x^3(x^2-1)^2(x^2-2)$
3	$\frac{1}{672000}x^4(x^2-1)^3(x^2-2^2)^2(x^2-3^2)$
4	$\frac{1}{102900000}x^5(x^2-1)^4(x^2-2^2)^3(x^2-3^2)^2(x^2-4^2)$

First Result

Theorem (K. Dilcher and LJ, 2020)

$$H_n\left(B_{2k+1}\left(\frac{x+1}{2}\right)\right) = (-1)^{\binom{n+1}{2}}\left(\frac{x}{2}\right)^{n+1}\prod_{\ell=1}^{n}\left(\frac{\ell^4(x^2-\ell^2)}{4(2\ell+1)(2\ell-1)}\right)^{n+1-\ell}.$$

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Theorem (K. Dilcher and LJ, 2020)

$$\sum_{k=0}^{\infty} B_{2k+1}\left(\tfrac{x+1}{2}\right)z^{2k} = \frac{1}{2z^2}\left(\psi'\left(\tfrac{1}{z}+\tfrac{1-x}{2}\right)-\psi'\left(\tfrac{1}{z}+\tfrac{1+x}{2}\right)\right)$$

$$= \cfrac{\frac{x}{2}}{1+\sigma_0 z^2+\cfrac{\tau_1 z^4}{1+\sigma_1 z^2+\cfrac{\tau_2 z^4}{1+\sigma_2 z^2+\ddots}}},$$

where $\psi' = (\log\Gamma)''$, $\sigma_n = \binom{n+1}{2} - \frac{x^2-1}{4}$, *and* $\tau_n = \frac{n^4(x^2-n^2)}{4(2n+1)(2n-1)}$.

Orthogonal Polynomials

Theorem

Given a sequence $c = (c_0, c_1, \ldots)$, *if* μ *is the measure such that* $c_k = \int_{\mathbb{R}} y^k d\mu(y)$, *then there exists a unique sequence of monic polynomials* $P_n(y)$ *of degree* n, *such that*

$$\int_{\mathbb{R}} P_m(y) P_n(y) d\mu(y) = \zeta_n \delta_{m,n}$$

for some sequence of constants ζ_n.

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for some sequence of constants ζ_n*. In addition,*

$$P_n(y) = \frac{1}{H_{n-1}(\mathrm{c})} \det \begin{pmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-1} \\ 1 & y & \cdots & y^n \end{pmatrix}.$$

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If P_n*'s three-term recurrence is given by*

$$P_{n+1}(y) = (y + s_n) P_n(y) - t_n P_{n-1}(y),$$

then

$$\sum_{k=0}^{\infty} c_k z^k = \cfrac{c_0}{1 + s_0 z - \cfrac{t_1 z^2}{1 + s_1 z - \cfrac{t_2 z^2}{1 + s_2 z - \ddots}}} \quad \text{and} \quad H_n(c_k) = c_0^{n+1} t_1^n t_2^{n-1} \cdots t_n$$

Key Step

$$\sum_{k=0}^{\infty} B_{2k+1}\left(\tfrac{x+1}{2}\right) z^{2k} = \frac{1}{2z^2}\left(\psi'\left(\tfrac{1}{z} + \tfrac{1-x}{2}\right) - \psi'\left(\tfrac{1}{z} + \tfrac{1+x}{2}\right)\right)$$

$$= \cfrac{\frac{x}{2}}{1 + \sigma_0 z^2 + \cfrac{\tau_1 z^4}{1 + \sigma_1 z^2 + \cfrac{\tau_2 z^4}{1 + \sigma_2 z^2 + \ddots}}}$$

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$$\sum_{k=0}^{\infty} B_{2k+1}\left(\tfrac{x+1}{2}\right) z^{2k} = \frac{1}{2z^2}\left(\psi'\left(\tfrac{1}{z}+\tfrac{1-x}{2}\right)-\psi'\left(\tfrac{1}{z}+\tfrac{1+x}{2}\right)\right)$$

$$= \cfrac{\frac{x}{2}}{1+\sigma_0 z^2+\cfrac{\tau_1 z^4}{1+\sigma_1 z^2+\cfrac{\tau_2 z^4}{1+\sigma_2 z^2+\ddots}}}$$

$$\sum_{k=0}^{\infty} \frac{1}{(s-b+2k+1)^2} - \sum_{k=0}^{\infty} \frac{1}{(s+b+2k+1)^2}$$

$$= \cfrac{b}{1(s^2-b^2+1)-\cfrac{4(1^2-b^2)1^4}{3(s^2-b^2+5)-\cfrac{4(2^2-b^2)2^4}{5(s^2-b^2+13)-\ddots}}}.$$

and

$$\psi'(z) = \sum_{k=0}^{\infty} \frac{1}{(z+k)^2}$$

Catalan Numbers Again

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Contractions

- Odd

$$\cfrac{1}{1-\cfrac{\alpha_1 t}{1-\cfrac{\alpha_2 t}{1-\ddots}}} = 1 + \cfrac{\alpha_1 t}{1-(\alpha_1+\alpha_2)t-\cfrac{\alpha_2\alpha_3 t^2}{1-(\alpha_3+\alpha_4)t-\cfrac{\alpha_4\alpha_5 t^2}{1-\ddots}}}$$

- Even

$$\cfrac{1}{1-\cfrac{\alpha_1 t}{1-\cfrac{\alpha_2 t}{1-\ddots}}} = \cfrac{1}{1-\alpha_1 t-\cfrac{\alpha_1\alpha_2 t^2}{1-(\alpha_2+\alpha_3)t-\cfrac{\alpha_3\alpha_4 t^2}{1-\ddots}}}$$

Euler Polynomials

Definition

The Euler numbers E_n and Euler polynomials are defined by their exponential generating functions

$$\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \quad \text{and} \quad \frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

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Theorem (K. Dilcher and LJ, 2020)

For $\nu = 0, 1, 2,$

$$\sum_{k=0}^{\infty} E_{2k+\nu}\left(\tfrac{x+1}{2}\right) z^k = \cfrac{E_\nu\left(\frac{x+1}{2}\right)}{1 + \sigma_0^{(\nu)} z + \cfrac{\tau_1^{(\nu)} z^2}{1 + \sigma_1^{(\nu)} z + \cfrac{\tau_2^{(\nu)} z^2}{1 + \sigma_2^{(\nu)} z + \ddots}}}$$

where $\sigma_n^{(\nu)} = (2n+1)(n+\frac{\nu}{2}) - \frac{x^2-1}{4}$ *and* $\tau_n^{(\nu)} = n^2(x^2 - (2n+\nu-1)^2)/4.$

Left-shifts

Theorem

Given a sequence c_k and the corresponding orthogonal monic polynomials, with 3-term recurrence

$$P_{n+1}(y) = (y + s_n)P_n(y) - t_n P_{n-1}(y),$$

$$H_n(c_{k+1}) = H_n(c_k) \cdot \det \begin{pmatrix} -s_0 & 1 & 0 & 0 & \cdots & 0 \\ t_1 & -s_1 & 1 & 0 & \cdots & 0 \\ 0 & t_2 & -s_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & t_n & -s_n \end{pmatrix}$$

and $H_n(b_{k+2}) = H_n(b_k) \cdot D_n$ for some expression D_n.

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and $H_n(b_{k+2}) = H_n(b_k) \cdot D_n$ for some expression D_n.

$$H_n(B_k) = (-1)^{\binom{n+1}{2}} \prod_{\ell=1}^{n} \left(\frac{\ell^4}{4(2\ell+1)(2\ell-1)} \right)^{n+1-\ell}$$

$$H_n(B_{k+1}) = \frac{(-1)^{\binom{n+1}{2}}}{2^{n+1}} \prod_{\ell=1}^{n} \left(\frac{\ell^2(\ell+1)^2}{4(2\ell+1)^2} \right)^{n+1-\ell}$$

$$H_n(B_{k+2}) = \frac{(-1)^{\binom{n+1}{2}}}{6^{n+1}} \prod_{\ell=1}^{n} \left(\frac{\ell(\ell+1)^2(\ell+2)}{4(2\ell+1)(2\ell+3)} \right)^{n+1-\ell}$$

Results (K.~Dilcher and LJ, 2020)

$$B_{2k+1}\left(\frac{x+1}{2}\right), E_{2k}\left(\frac{x+1}{2}\right), E_{2k+1}\left(\frac{x+1}{2}\right), E_{2k+2}\left(\frac{x+1}{2}\right),$$

$$B_k\left(\frac{x+r}{q}\right) - B_k\left(\frac{x+s}{q}\right), E_k\left(\frac{x+r}{q}\right) \pm E_k\left(\frac{x+s}{q}\right),$$

$$kE_{k-1}(x), B_{k+1,\chi_{8,1}}(x), B_{k+1,\chi_{8,2}}(x), B_{k+1,\chi_{12,1}}(x), B_{k+1,\chi_{12,2}}(x),$$

$$(2k+1)E_{2k}, (2^{2k+2}-1)B_{2k+2}, (2k+1)B_{2k}\left(\frac{1}{2}\right), (2k+3)B_{2k+2},$$

Derivatives:

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Lemma (K. Dilcher and LJ, 2020)

Let $c_k(x)$ be a sequence of C^1 functions, and let $P_n(y;x)$ be the corresponding monic orthogonal polynomials. If $c_k(x_0) = 0$ for some $x_0 \in \mathbb{C}$ and for all $k \geq 0$, then $P_n(y;x_0)$ are the monic orthogonal polynomials with respect to the sequence of derivatives $c_k'(x_0)$, as long as $H_n(c_k'(x_0))$ are all nonzero.

Derivatives:

Lemma (K. Dilcher and LJ, 2020)

Let $c_k(x)$ be a sequence of C^1 functions, and let $P_n(y;x)$ be the corresponding monic orthogonal polynomials. If $c_k(x_0)=0$ for some $x_0\in\mathbb{C}$ and for all $k\geq 0$, then $P_n(y;x_0)$ are the monic orthogonal polynomials with respect to the sequence of derivatives $c_k'(x_0)$, as long as $H_n(c_k'(x_0))$ are all nonzero.

$(2k+1)E_{2k}$

$$E_{2k+1}\left(\frac{x+1}{2}\right)\longleftrightarrow P_{n+1}=\left(y+(2n+1)\left(n+\frac{1}{2}\right)-\frac{x^2-1}{4}\right)P_n-\frac{n^2(x^2-4n^2)}{4}P_{n-1};$$

and recall that

$$E_k'(x)=kE_{k-1}(x)\quad\text{and}\quad E_{2k+1}\left(\frac{1}{2}\right)=0.$$

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Namely,

$$(b_0, b_1, \ldots) = (a, c_0, c_1, \ldots).$$

$$H_n(B_{2k}) = (-1)^n \frac{(4n+3)!}{(n+1) \cdot (2n+1)!^3} H_n(B_{2k+2}) \mathcal{H}_{2n+1},$$

for the harmonic numbers

$$\mathcal{H}_n = 1 + \frac{1}{2} + \cdots \frac{1}{n}.$$

Right-shifted

Lemma (K. Dilcher and LJ, 2020)

Let s_n and t_n be the sequences appearing in the 3-term recurrence of the monic orthogonal polynomial sequences for c_k. Then,

$$\frac{H_{n+1}(b_k)}{H_n(c_k)} = -s_n \frac{H_n(b_k)}{H_{n-1}(c_k)} - t_n \frac{H_{n-1}(b_k)}{H_{n-2}(c_k)}.$$

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$b_k,\ k \geq 1$	B_{k-1}	B_{2k}	$(2k+1)B_{2k}$	$(2^{2k}-1)B_{2k}$
b_0	0	1	1	0

$b_k,\ k \geq 1$	E_{2k-2}	$E_{k-1}(1)$	$E_{k+3}(1)$	$E_{2k-1}(1)$	$E_{2k+5}(1)$	$\frac{E_k(1)}{k!}$
b_0	0	0	$-\frac{1}{4}$	0	$\frac{1}{2}$	1

$b_k,\ k \geq 1$	$\frac{E_{2k-1}(1)}{(2k-1)!}$	$E_{2k-2}\left(\frac{x+1}{2}\right)$	$(2k+1)E_{2k}$		
b_0	0	0	0		

The last one

$$I_k := \sum_{c=1}^{r} c^k \quad \text{and} \quad V_n := \begin{pmatrix} I_0 & I_2 & \cdots & I_{2n} \\ I_2 & I_4 & \cdots & I_{2n+2} \\ \vdots & \vdots & \ddots & \vdots \\ I_{2n} & I_{2n+2} & \cdots & I_{4n} \end{pmatrix}.$$

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Remark

It is known that $(V_{r-1}^{-1})_{1,1} = 2\left(\binom{4r}{2r}/\binom{2r}{r}^2 - 1\right)$.

Results

Proposition (LJ and Y. Li, 2021)

$$H_n\left(\frac{B_{2k+1}\left(\frac{x+1}{2}\right)}{2k+1}\right) = \left(\frac{x}{2}\right)^{n+1} \prod_{\ell=1}^{n} \left(\frac{(2\ell)^2(2\ell-1)^2(x^2-(2\ell-1)^2)(x^2-(2\ell)^2)}{16(4\ell-3)(4\ell-1)^2(4\ell+1)}\right)^{n+1-\ell}$$

$$H_n\left(\frac{B_{2k+3}\left(\frac{x+1}{2}\right)}{2k+3}\right) = \left(\frac{x^3-x}{24}\right)^{n+1} \prod_{\ell=1}^{n} \left(\frac{(2\ell)^2(2\ell+1)^2(x^2-(2\ell+1)^2)(x^2-(2\ell)^2)}{16(4\ell-3)(4\ell+1)^2(4\ell+3)}\right)^{n+1-\ell}$$

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Theorem (Christian Krattenthaler)

Let $(a)_n = a(a+1)\cdots(a+n-1)$ be the Pochhammer symbol.

$$H_n\left(\frac{B_{2k+5}\left(\frac{x+1}{2}\right)}{2k+5}\right)=\frac{1}{5\cdot 2^{n+2}}\prod_{i=1}^{n}\frac{(2i+3)!^2(2i+2)!^2}{(4i+5)!(4i+4)!}\prod_{\ell=0}^{n}(x-2n-1+2\ell)_{4n-4\ell+3}$$

$$\times\sum_{i=1}^{n+2}\frac{(2i-1)\left(n+\frac{5}{2}\right)_{i-1}\left(\frac{x+1}{2}\right)_{n+2}\left(\frac{x}{2}-n-\frac{3}{2}\right)_{n+2}}{\left(n-i+\frac{5}{2}\right)_i(n+2-i)!(n+1+i)!(x^2-(2i-1)^2)}$$

$$\det V_n = 2^{2n^2-2n-1}\prod_{i=1}^{n}\frac{(2i)!^4}{(4i)!(4i+1)!}\prod_{\ell=0}^{n}(r-\ell)_{2\ell+1}\prod_{\ell=0}^{n-1}\left(r+\frac{1}{2}-\ell\right)_{2\ell+1}$$

$$\times\sum_{i=1}^{n+1}\frac{(2n+2i)!(2n+2-2i)!(r+1)_{n+1}}{(n+i)!^2(n+1-i)!^2(r+i)}.$$

Stirling Numbers

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Let $s(n,k)$ and $S(n,k)$ be the Stirling numbers of the first and second kinds, respective. For any $r \in \mathbb{N}$ and $j = 0, 1, 2 \ldots, r$, we have

$$\sum_{k=0}^{r} \frac{1}{2j+2k+1} \left(\sum_{m=0}^{2j+2k} \binom{2j+2k+1}{m} \left((r+1)^{2j+2k+1-m} - 1 \right) \right.$$
$$\times \sum_{\ell=0}^{m} \frac{(-1)^{\ell} \ell!}{\ell+1} S(m,\ell)$$
$$\left. \times \sum_{i=0}^{2k+2} (-1)^{r+1+i} s(r+1,i) s(r+1,2k+2-i) \right) = 0.$$

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Thank you!