Bernoulli Symbol and Multiple Zeta Function at Non-negative Integers

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第一届多重 zeta 值及相关领域国际研讨会

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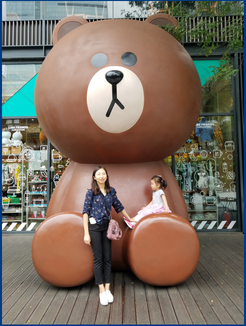

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Tianjin University

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Bernoulli Polynomials, Numbers

Definition

Bernoulli numbers B_n and Bernoulli polynomials $B_n(x)$ are defined by their generating functions

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad \text{and} \quad \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

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$$B_1 = -\frac{1}{2} \quad \text{and} \quad B_{2n+1} = 0 \Rightarrow \quad \zeta(-m) = -\frac{(-1)^m B_{m+1}}{m+1}, m = 0, 1, 2, \ldots$$

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$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \mathcal{B}^k x^{n-k} = (\mathcal{B} + x)^n.$$

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Theorem (A. Dixit, V. H. Moll, and C. Vignat)

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$$B_n(x) = (\mathcal{B}+x)^n = \frac{\pi}{2} \int_{\mathbb{R}} \left(x + \imath t - \frac{1}{2}\right)^n \operatorname{sech}^2(\pi t)\, dt.$$

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$$\beta^n = \frac{B_n}{n}.$$

MZV and Analytic Continuation

Theorem (B. Sadaoui, 2014)

Based on Raabe's identity, and by linking

$$Y_a(n) = \int_{[1,\infty)^r} \frac{dx}{(x_1+a_1)\cdots(x_1+a_1+\cdots+x_r+a_r)^{n_r}}$$

$$Z(n,z) = \sum_{k_1,\ldots,k_r>0} \frac{1}{(k_1+z_1)^{n_1}\cdots(k_1+z_1+\cdots+k_r+z_r)^{n_r}}$$

by

$$Y_0(n) = \int_{[0,1]^r} Z(n,z)\,dz,$$

for positive integers $n_1,\ldots,n_r$,

$$\zeta_r(-n_1,\ldots,-n_r) = (-1)^r \sum_{k_2,\ldots,k_r} \frac{1}{\bar{n}+r-\bar{k}} \prod_{j=2}^{r} \frac{\binom{\sum_{i=j}^{r} n_i - \sum_{i=j+1}^{n} k_i + r - j + 1}{k_j}}{\sum_{i=j}^{r} n_i - \sum_{i=j}^{n} k_i + r - j + 1}$$

$$\times \sum_{l_1,\ldots,l_r} \binom{\bar{n}+r-\bar{k}}{l_1}\binom{k_2}{l_2}\cdots\binom{k_r}{l_r} B_{l_1}\cdots B_{l_r},$$

$\bar{n} = \sum_{j=1}^{n} n_j$, $\bar{k} = \sum_{j=2}^{r} k_j$, $k_2,\ldots k_r \geq 0$, $l_j \leq k_j$ *for* $2 \leq j \leq r$ *and* $l_1 \leq \bar{n}+r+\bar{k}$.

MZV and the $\mathcal{C}$-symbol

Theorem (LJ, V. H. Moll, and C. Vignat)

$$\zeta_r\left(-n_1,\ldots,-n_r\right)=\prod_{k=1}^{r}\left(-1\right)^{n_k}\mathcal{C}_{1,\ldots,k}^{n_k+1},$$

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$\zeta_1(-n)$

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for positive integers n_1 and n_2,

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When $n_1 + n_2$ is odd, the Bernoulli number $B_{n_1+n_2+2} = 0$, so that the quasi-shuffle identity holds as expected, since the depth-2 multiple zeta function is holomorphic at these points.

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Sadaoui's analytic continuation used does not preserve the quasi-shuffle identity, but the correction term is simple.

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When $n_1 + n_2$ is odd, the Bernoulli number $B_{n_1+n_2+2} = 0$, so that the quasi-shuffle identity holds as expected, since the depth-2 multiple zeta function is holomorphic at these points.

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Theorem (S.Akiyama and Y.Tanigawa, 2002)

By using Euler-Maclaurin summation formula, for positive integers $n_1, \dots, n_r$,

$$\begin{aligned}\zeta_r(-n_1,\dots,-n_r) &= -\frac{\zeta_{r-1}(-n_1,\dots,-n_{r-2},-n_{r-1}-n_r-1)}{1+n_r} \\ &\quad -\frac{\zeta_{r-1}(-n_1,\dots,-n_{r-2},-n_{r-1}-n_r)}{2} \\ &\quad +\sum_{q=1}^{n_r}(-n_r)_q\, a_q \zeta_{r-1}(-n_1,\dots,-n_{r-2},-n_{r-1}-n_r+q),\end{aligned}$$

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where $a_q = B_{q+1} / (q+1)!$.

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Harmonic Sums

Definition

$$S_{a_1,\dots,a_k}(N) = \sum_{N \geq n_1 \geq \dots \geq n_k \geq 1} \frac{\mathrm{sgn}(a_1)^{n_1}}{n_1^{|a_1|}} \times \dots \times \frac{\mathrm{sgn}(a_k)^{n_k}}{n_k^{|a_k|}}, \quad N \in \mathbb{N}$$

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Random Walk

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- one can only jump to sites that are NOT to the right of the current site, with equal probabilities;
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$$\begin{matrix} \bullet & \bullet & \bullet & \cdots & \bullet & \bullet \\ 1 & 2 & 3 & & N-1 & N \end{matrix}$$

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$$\mathbb{P}(n_{k+1} = 1) = \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{N n_1 \cdots n_k} = \frac{S_{\underbrace{1, \ldots, 1}_{k}}(N)}{N}.$$

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$$\lim_{k \to \infty} S_{\underbrace{1, 1, \ldots, 1}_{k}}(N) = N \Leftrightarrow \lim_{k \to \infty} \mathbb{P}(n_{k+1} = 1) = 1.$$

Matrix Representations

Theorem (**LJ**, and D. Y. Shi)

Define

$$\mathcal{S}\left(f_1,\ldots,f_k;N,m\right) := \sum_{N\geq n_1\geq\cdots\geq n_k\geq m} f_1(n_1)\cdots f_k\left(n_k\right),$$

$$\mathcal{A}\left(f_1,\ldots,f_k;k;N,m\right) := \sum_{N> n_1>\cdots> n_k\geq m} f_1(n_1)\cdots f_k\left(n_k\right),$$

and three matrices

$$\mathbf{P}_N = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}, \quad \mathbf{S}_{N|f_l} := \begin{pmatrix} f_l(1) & 0 & 0 & \cdots & 0 \\ f_l(2) & f_l(2) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_l(N) & f_l(N) & f_l(N) & \cdots & f_l(N) \end{pmatrix}$$

and

$$\mathbf{A}_{N|f_l} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ f_l(1) & 0 & 0 & \cdots & 0 & 0 \\ f_l(2) & f_l(2) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_l(N-1) & f_l(N-1) & f_l(N-1) & \cdots & f_l(N-1) & 0 \end{pmatrix}.$$

Then,

$$\mathcal{S}\left(f_1,\ldots,f_k;N,m\right) = \left(\mathbf{P}_N \cdot \prod_{l=1}^{k} \mathbf{S}_{N|f_l}\right)_{N,m} \text{ and } \mathcal{A}\left(f_1,\ldots,f_k;N,m\right) = \left(\mathbf{P}_N \cdot \prod_{l=1}^{k} \mathbf{A}_{N|f_l}\right)_{N,m}.$$

$N \to \infty$

$$S(i_1, \ldots, i_k) := \mathcal{S}\left(\frac{1}{x^{i_1}}, \ldots, \frac{1}{x^{i_k}}; \infty, 1\right) = \sum_{n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1^{i_1} \cdots n_k^{i_k}}$$

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Theorem (LJ, and D. Y. Shi)

$$S(f,g;N-1,m) = A(f,g;N,m) + A(fg;N,m)$$

$$S(f,g,h;N-1,m) = A(f,g,h;N,m) + A(fg,h;N,m) + A(f,gh;N,M) + A(fgh;N,m)$$

$\mathcal{V}$ and $\mathcal{H}$ Symbols

Definition

The $\mathcal{H}$ symbol is defined by

$$(\mathcal{H}(N))^n := 1^n + 2^n + \cdots + (N-1)^n =: H_{-n}(N),$$

where

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$$H_{-n_1,\ldots,-n_r}(N) = \prod_{k=1}^{r} \mathcal{H}_{1,\ldots,k}^{n_k} = \prod_{k=1}^{r} (\mathcal{B}_k + \mathcal{V}_{1,\ldots,k})^{n_k}.$$

where $\mathcal{H}_1 = \mathcal{H}(N)$, $\mathcal{V}_1 = \mathcal{V}(N)$ *and recursively* $\mathcal{H}_{1,\ldots,k} = \mathcal{H}(\mathcal{H}_{1,\ldots,k-1})$ *and* $\mathcal{V}_{1,\ldots,k} = \mathcal{V}(\mathcal{B}_{k-1} + \mathcal{V}_{1,\ldots,k-1})$.

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$$\lim_{N\to\infty} \mathcal{H} = \mathcal{C}.$$

End

Thank you for your listening!