Random Walk Model on Finite Number of Sites

Lin Jiu

DUKE KUNSHAN
Zu Chongzhi Center for Mathematics and Computational Sciences

安徽大學
ANHUI UNIVERSITY 1928

October 19th, 2022

Acknowledgment

Dr. Victor H. Moll

Dr. Christophe Vignat

Dr. Italo Simonelli

Heng Yue

Motivation

Motivation

Definition

The **Euler numbers** E_n, **Euler polynomials** $E_n(x)$, and **Euler polynomials of order** p, $E_n^{(p)}(x)$, are defined via their (exponential) generating functions

$$\frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad \frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad \left(\frac{2}{e^t+1}\right)^p e^{xt} = \sum_{n=0}^{\infty} E_n^{(p)}(x) \frac{t^n}{n!}.$$

Motivation

Definition

The **Euler numbers** E_n, **Euler polynomials** $E_n(x)$, and **Euler polynomials of order** p, $E_n^{(p)}(x)$, are defined via their (exponential) generating functions

$$\frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad \frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad \left(\frac{2}{e^t+1}\right)^p e^{xt} = \sum_{n=0}^{\infty} E_n^{(p)}(x) \frac{t^n}{n!}.$$

Example

$E_n^{(1)}(x) = E_n(x)$ and $E_n = 2^n E_n(1/2)$.

Motivation

Definition

The **Euler numbers** E_n, **Euler polynomials** $E_n(x)$, and **Euler polynomials of order** p**,** $E_n^{(p)}(x)$, are defined via their (exponential) generating functions

$$\frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad \frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad \left(\frac{2}{e^t+1}\right)^p e^{xt} = \sum_{n=0}^{\infty} E_n^{(p)}(x) \frac{t^n}{n!}.$$

Example

$E_n^{(1)}(x) = E_n(x)$ and $E_n = 2^n E_n(1/2)$.

Fact: Convolution

$$E_n^{(p)}(x) = \sum_{k_1+\cdots+k_p=n} \binom{n}{k_1,\ldots,k_p} E_{k_1}(x) E_{k_2}(0) \cdots E_{k_p}(0).$$

Inverse Problem

Inverse Problem

Problem

$$E_n(x) = P\left(E_{n_1}^{(p_1)}(x), \ldots, E_{n_k}^{(p_k)}(x)\right)?$$

Inverse Problem

Problem

$$E_n(x) = P\left(E_{n_1}^{(p_1)}(x), \ldots, E_{n_k}^{(p_k)}(x)\right)?$$

Theorem(**LJ**, V. H. Moll and C. Vignat, 2014)

For any positive integer N,

$$E_n(x) = \frac{1}{N^n} \sum_{\ell=N}^{\infty} p_\ell^{(N)} E_n^{(\ell)} \left(\frac{\ell - N}{2} + Nx \right),$$

Inverse Problem

Problem

$$E_n(x) = P\left(E_{n_1}^{(p_1)}(x), \ldots, E_{n_k}^{(p_k)}(x)\right)?$$

Theorem(**LJ**, V. H. Moll and C. Vignat, 2014)

For any positive integer N,

$$E_n(x) = \frac{1}{N^n} \sum_{\ell=N}^{\infty} p_\ell^{(N)} E_n^{(\ell)}\left(\frac{\ell - N}{2} + Nx\right),$$

where

$$\frac{1}{T_N(1/z)} = \sum_{\ell=0}^{\infty} p_\ell^{(N)} z^\ell,$$

Inverse Problem

Problem

$$E_n(x) = P\left(E_{n_1}^{(p_1)}(x), \ldots, E_{n_k}^{(p_k)}(x)\right)?$$

Theorem(**LJ**, V. H. Moll and C. Vignat, 2014)

For any positive integer N,

$$E_n(x) = \frac{1}{N^n} \sum_{\ell=N}^{\infty} p_\ell^{(N)} E_n^{(\ell)}\left(\frac{\ell - N}{2} + Nx\right),$$

where

$$\frac{1}{T_N(1/z)} = \sum_{\ell=0}^{\infty} p_\ell^{(N)} z^\ell, \quad T_N(\cos\theta) = \cos(N\theta)$$

Inverse Problem

Problem

$$E_n(x) = P\left(E_{n_1}^{(p_1)}(x), \ldots, E_{n_k}^{(p_k)}(x)\right)?$$

Theorem(**LJ**, V. H. Moll and C. Vignat, 2014)

For any positive integer N,

$$E_n(x) = \frac{1}{N^n} \sum_{\ell=N}^{\infty} p_\ell^{(N)} E_n^{(\ell)}\left(\frac{\ell - N}{2} + Nx\right),$$

where

$$\frac{1}{T_N(1/z)} = \sum_{\ell=0}^{\infty} p_\ell^{(N)} z^\ell, \;\; T_N(\cos\theta) = \cos(N\theta)$$

Namely, T_n is the Chebyshev polynomial of the 1st kind.

Example

$\underline{N = 2:}$ $T_2(z) = 2z^2 - 1$

Example

$\underline{N = 2:}$ $T_2(z) = 2z^2 - 1 \big(\cos(2\theta) = 2\cos^2\theta - 1\big)$

Example

<u>$N = 2$:</u> $T_2(z) = 2z^2 - 1 \big(\cos(2\theta) = 2\cos^2\theta - 1\big)$ and $\frac{1}{T_2(1/z)} = \frac{z^2}{2-z^2}$

Example

$\underline{N=2:}$ $T_2(z) = 2z^2 - 1 \big(\cos(2\theta) = 2\cos^2\theta - 1\big)$ and $\frac{1}{T_2(1/z)} = \frac{z^2}{2-z^2}$

$$E_n(x) = \frac{1}{2^n} \sum_{\ell=2}^{\infty} p_\ell^{(2)} E_n^{(\ell)} \left(\frac{\ell}{2} - 1 + 2x\right),$$

Example

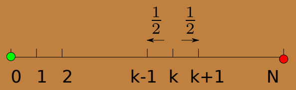
<u>$N = 2$:</u> $T_2(z) = 2z^2 - 1 \big(\cos(2\theta) = 2\cos^2\theta - 1\big)$ and $\frac{1}{T_2(1/z)} = \frac{z^2}{2-z^2}$

$$E_n(x) = \frac{1}{2^n} \sum_{\ell=2}^{\infty} p_\ell^{(2)} E_n^{(\ell)} \left(\frac{\ell}{2} - 1 + 2x \right), \; p_\ell^{(2)} = \begin{cases} \frac{1}{2^k}, & \text{if } \ell = 2k; \\ 0, & \text{otherwise} \end{cases}$$

Example

$\underline{N=2:}$ $T_2(z) = 2z^2 - 1 \big(\cos(2\theta) = 2\cos^2\theta - 1\big)$ and $\frac{1}{T_2(1/z)} = \frac{z^2}{2-z^2}$

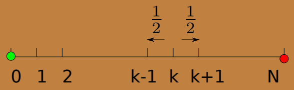
$$E_n(x) = \frac{1}{2^n}\sum_{\ell=2}^{\infty} p_\ell^{(2)} E_n^{(\ell)}\left(\frac{\ell}{2} - 1 + 2x\right), \quad p_\ell^{(2)} = \begin{cases} \frac{1}{2^k}, & \text{if } \ell = 2k; \\ 0, & \text{otherwise} \end{cases}$$



Example

$\underline{N=2:}$ $T_2(z) = 2z^2 - 1\big(\cos(2\theta) = 2\cos^2\theta - 1\big)$ and $\frac{1}{T_2(1/z)} = \frac{z^2}{2-z^2}$

$$E_n(x) = \frac{1}{2^n}\sum_{\ell=2}^{\infty} p_\ell^{(2)} E_n^{(\ell)}\left(\frac{\ell}{2} - 1 + 2x\right), \quad p_\ell^{(2)} = \begin{cases} \frac{1}{2^k}, & \text{if } \ell = 2k; \\ 0, & \text{otherwise} \end{cases}$$

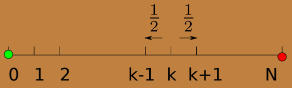


- 0 is the source and N is the sink;

Example

$\underline{N = 2:}$ $T_2(z) = 2z^2 - 1 \big(\cos(2\theta) = 2\cos^2\theta - 1\big)$ and $\frac{1}{T_2(1/z)} = \frac{z^2}{2-z^2}$

$$E_n(x) = \frac{1}{2^n}\sum_{\ell=2}^{\infty} p_\ell^{(2)} E_n^{(\ell)}\left(\frac{\ell}{2} - 1 + 2x\right), \; p_\ell^{(2)} = \begin{cases} \frac{1}{2^k}, & \text{if } \ell = 2k; \\ 0, & \text{otherwise} \end{cases}$$

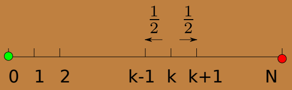


- 0 is the source and N is the sink;
- at each $k = 1, \ldots, N-1$, it is a "fair coin" walk;

Example

<u>$N = 2$:</u> $T_2(z) = 2z^2 - 1 \big(\cos(2\theta) = 2\cos^2\theta - 1\big)$ and $\frac{1}{T_2(1/z)} = \frac{z^2}{2-z^2}$

$$E_n(x) = \frac{1}{2^n}\sum_{\ell=2}^{\infty} p_\ell^{(2)} E_n^{(\ell)}\left(\frac{\ell}{2} - 1 + 2x\right), \quad p_\ell^{(2)} = \begin{cases} \frac{1}{2^k}, & \text{if } \ell = 2k; \\ 0, & \text{otherwise} \end{cases}$$



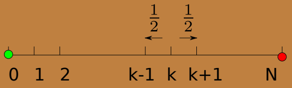
- 0 is the source and N is the sink;
- at each $k = 1, \ldots, N-1$, it is a "fair coin" walk;
- let ν_N be the random number of steps for this process.

$$p_\ell^{(N)} = \mathbb{P}(\nu_N = \ell)$$

Example

$\underline{N=2:}$ $T_2(z) = 2z^2 - 1 \big(\cos(2\theta) = 2\cos^2\theta - 1\big)$ and $\frac{1}{T_2(1/z)} = \frac{z^2}{2-z^2}$

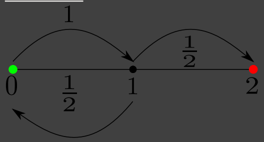
$$E_n(x) = \frac{1}{2^n}\sum_{\ell=2}^{\infty} p_\ell^{(2)} E_n^{(\ell)}\left(\frac{\ell}{2} - 1 + 2x\right), \quad p_\ell^{(2)} = \begin{cases} \frac{1}{2^k}, & \text{if } \ell = 2k; \\ 0, & \text{otherwise} \end{cases}$$



- 0 is the source and N is the sink;
- at each $k = 1, \ldots, N-1$, it is a "fair coin" walk;
- let ν_N be the random number of steps for this process.

$$p_\ell^{(N)} = \mathbb{P}(\nu_N = \ell)$$

$\underline{N=2:}$



$$p_\ell^{(2)} = \begin{cases} \frac{1}{2^k}, & \text{if } \ell = 2k; \\ 0, & \text{otherwise} \end{cases}$$

Proof: Random Variable Approach

Proof: Random Variable Approach

1. Let $L \sim \operatorname{sech}(\pi t)$, then the Euler polynomial is given by

$$E_n(x) = \mathbb{E}\left[\left(x + iL - \frac{1}{2}\right)^n\right] = \int_{\mathbb{R}} \left(x + it - \frac{1}{2}\right)^n \operatorname{sech}(\pi t)\, \mathrm{d}t.$$

Proof: Random Variable Approach

1. Let $L \sim \operatorname{sech}(\pi t)$, then the Euler polynomial is given by

$$E_n(x) \quad = \quad \mathbb{E}\left[\left(x + iL - \frac{1}{2}\right)^n\right] = \int_{\mathbb{R}} \left(x + it - \frac{1}{2}\right)^n \operatorname{sech}(\pi t)\,\mathrm{d}t.$$

$$E_n^{(p)}(x) = \mathbb{E}\left[\left(x + i\left(L_1 + \cdots + L_p\right) - \tfrac{p}{2}\right)^n\right], \text{ i. i. d. } \; L_j \sim L.$$

Proof: Random Variable Approach

1. Let $L \sim \operatorname{sech}(\pi t)$, then the Euler polynomial is given by

$$E_n(x) = \mathbb{E}\left[\left(x + iL - \frac{1}{2}\right)^n\right] = \int_{\mathbb{R}} \left(x + it - \frac{1}{2}\right)^n \operatorname{sech}(\pi t)\, \mathrm{d}t.$$

$$E_n^{(p)}(x) = \mathbb{E}\left[\left(x + i\left(L_1 + \cdots + L_p\right) - \tfrac{p}{2}\right)^n\right], \text{ i. i. d. } \; L_j \sim L.$$

2. Note that ν_N is an integer valued random variable **independent** of the L_j 's:

$$\mathbb{E}\left[z^{\nu_N}\right] = \frac{1}{T_N\left(\frac{1}{z}\right)}.$$

Proof: Random Variable Approach

1. Let $L \sim \operatorname{sech}(\pi t)$, then the Euler polynomial is given by

$$E_n(x) = \mathbb{E}\left[\left(x + iL - \frac{1}{2}\right)^n\right] = \int_{\mathbb{R}} \left(x + it - \frac{1}{2}\right)^n \operatorname{sech}(\pi t)\, \mathrm{d}t.$$

$$E_n^{(p)}(x) = \mathbb{E}\left[\left(x + i\left(L_1 + \cdots + L_p\right) - \tfrac{p}{2}\right)^n\right], \text{ i. i. d. } \quad L_j \sim L.$$

2. Note that ν_N is an integer valued random variable **independent** of the L_j 's:

$$\mathbb{E}\left[z^{\nu_N}\right] = \frac{1}{T_N\left(\frac{1}{z}\right)}.$$

Theorem (Klebanov et al.)

The random variable

$$Z_N = \frac{1}{N}\sum_{j=1}^{\nu_N} L_j$$

has the same hyperbolic secant distribution (as L_j's).

L. B. Klebanov, A. V. Kakosyan, S. T. Rachev, and G. Temnov. On a class of distributions stable under random summation. *J. Appl. Prob.*, **49** (2012), 303–318.

Proof: Random Variable Approach

Proof: Random Variable Approach

3.

$$L \sim \frac{1}{N} \sum_{j=1}^{\nu_N} L_j$$

Proof: Random Variable Approach

3.

$$L \sim \frac{1}{N}\sum_{j=1}^{\nu_N} L_j \quad \Rightarrow \quad x + iL - \frac{1}{2} \sim x + \left(\frac{1}{N}\sum_{j=1}^{\nu_N} iL_j\right) - \frac{1}{2}$$

Proof: Random Variable Approach

3.

$$
\begin{aligned}
L \sim \frac{1}{N}\sum_{j=1}^{\nu_N} L_j \quad \Rightarrow \quad & x + iL - \frac{1}{2} \sim x + \left(\frac{1}{N}\sum_{j=1}^{\nu_N} iL_j\right) - \frac{1}{2} \\
\sim \quad & \frac{1}{N}\sum_{j=1}^{\nu_N}\left(iL_j - \frac{\nu_N}{2} + Nx - \frac{N}{2} + \frac{\nu_N}{2}\right)
\end{aligned}
$$

Proof: Random Variable Approach

3.

$$L \sim \frac{1}{N}\sum_{j=1}^{\nu_N} L_j \quad \Rightarrow \quad x + iL - \frac{1}{2} \sim x + \left(\frac{1}{N}\sum_{j=1}^{\nu_N} iL_j\right) - \frac{1}{2}$$

$$\sim \quad \frac{1}{N}\sum_{j=1}^{\nu_N}\left(iL_j - \frac{\nu_N}{2} + Nx - \frac{N}{2} + \frac{\nu_N}{2}\right)$$

Taking moments:

- LHS: $\mathbb{E}\left[\left(x + iL - \frac{1}{2}\right)^n\right] = E_n(x)$;

Proof: Random Variable Approach

3.

$$L \sim \frac{1}{N}\sum_{j=1}^{\nu_N} L_j \quad \Rightarrow \quad x + iL - \frac{1}{2} \sim x + \left(\frac{1}{N}\sum_{j=1}^{\nu_N} iL_j\right) - \frac{1}{2}$$

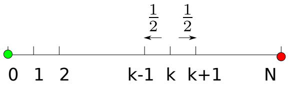
$$\sim \quad \frac{1}{N}\sum_{j=1}^{\nu_N}\left(iL_j - \frac{\nu_N}{2} + Nx - \frac{N}{2} + \frac{\nu_N}{2}\right)$$

Taking moments:

- LHS: $\mathbb{E}\left[\left(x + iL - \frac{1}{2}\right)^n\right] = E_n(x)$;
- RHS: Each $\nu_N = \ell$, with probability $p_\ell^{(N)}$ and
$$\mathbb{E}\left[\left(i\sum_{j=1}^{\ell} L_j - \tfrac{\ell}{2} + Nx - \tfrac{N}{2} + \tfrac{\ell}{2}\right)^n\right] = E_n^{(\ell)}\left(\tfrac{\ell - N}{2} + Nx\right).$$

□

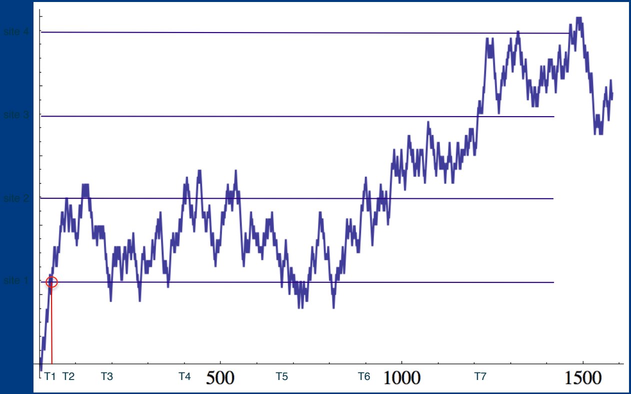
Reflected Brownian Motion



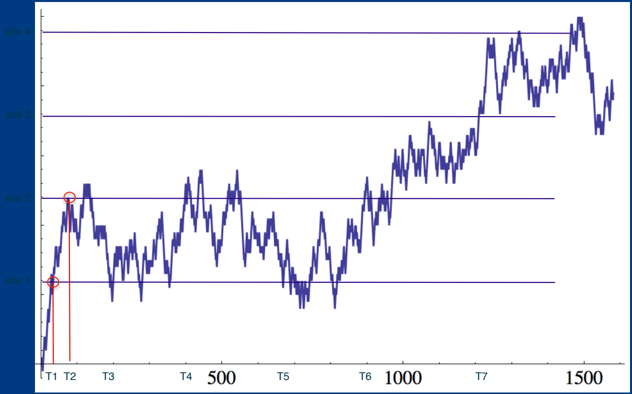
- 0 is the source and N is the sink;
- at each $k = 1, \ldots, N-1$, it is a "fair coin" walk;
- let ν_N be the random number of steps for this process.

$$p_\ell^{(N)} = \mathbb{P}(\nu_N = \ell)$$

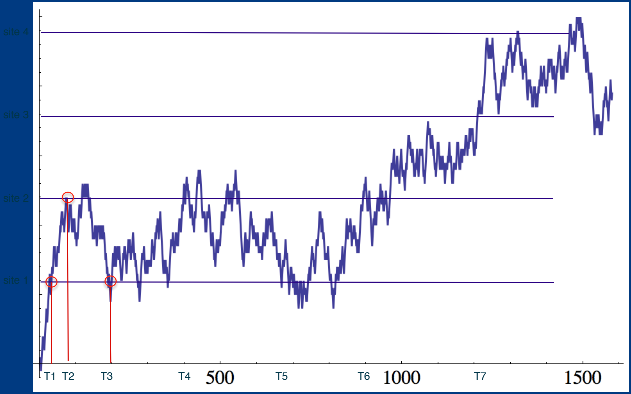
Reflected Brownian Motion



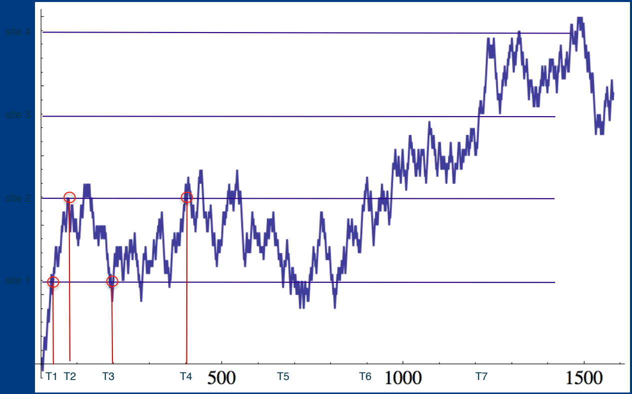
Reflected Brownian Motion



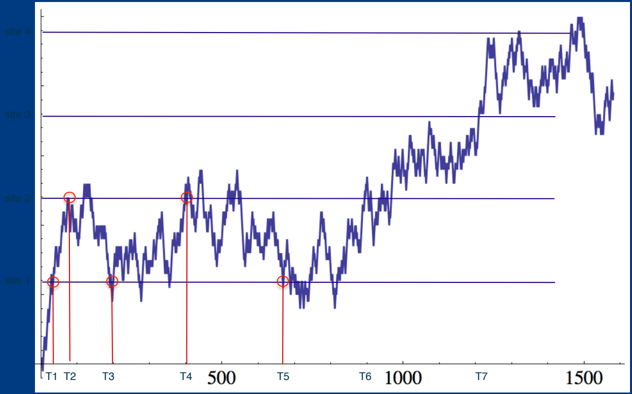
Reflected Brownian Motion



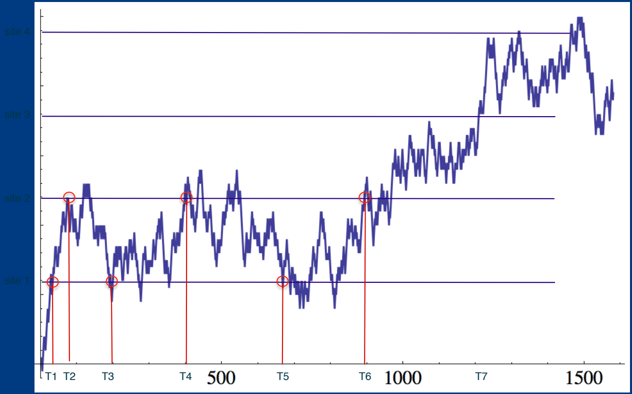
Reflected Brownian Motion



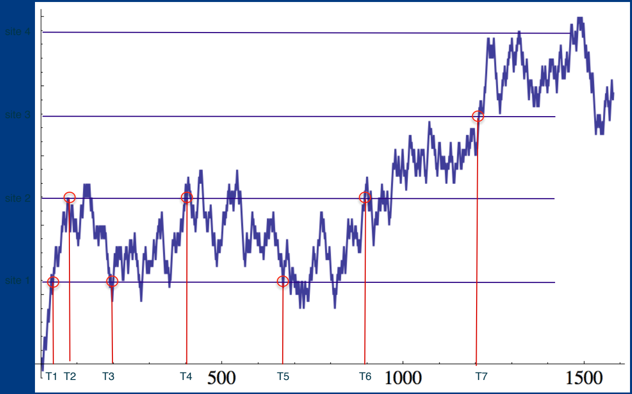
Reflected Brownian Motion



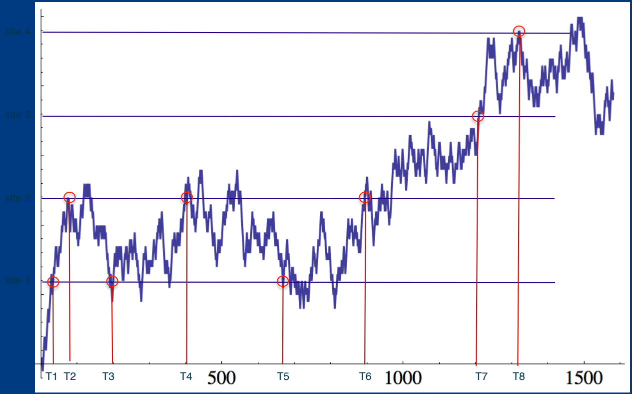
Reflected Brownian Motion



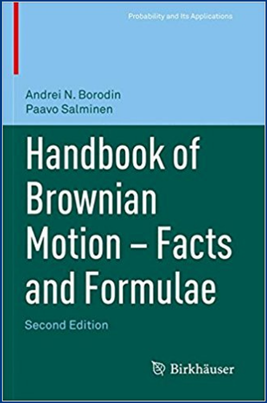
Reflected Brownian Motion



Reflected Brownian Motion

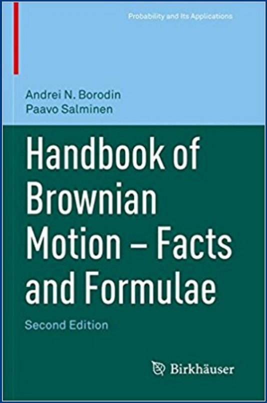


Hitting Time



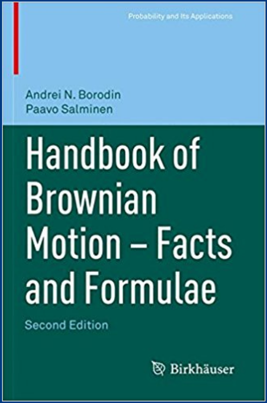
Hitting Time

- Reflected Brownian Motion on $\mathbb{R}_+$: W_t =distance to 0 at time t.



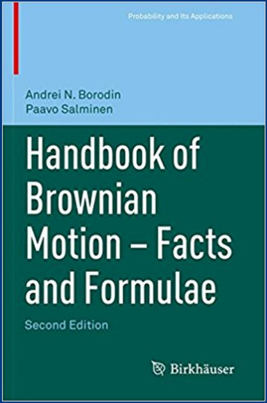
Hitting Time

- Reflected Brownian Motion on $\mathbb{R}_+$: W_t =distance to 0 at time t.
- Hitting times: $H_z := \min_t \{W_t = z\}$.



Hitting Time

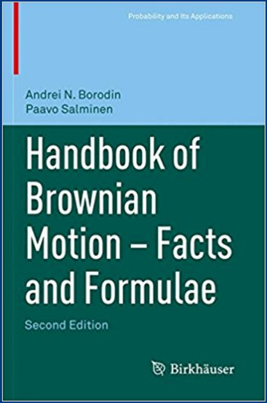
- Reflected Brownian Motion on $\mathbb{R}_+$: W_t =distance to 0 at time t.
- Hitting times: $H_z := \min_t \{W_t = z\}$.
- $$\mathbb{E}_x\left[e^{-\alpha H_z}\right] = \begin{cases} \frac{\cosh(xw)}{\cosh(zw)}, & 0 \leq x \leq z; \\ e^{-(x-z)w}, & z \leq x. \end{cases}$$



Hitting Time

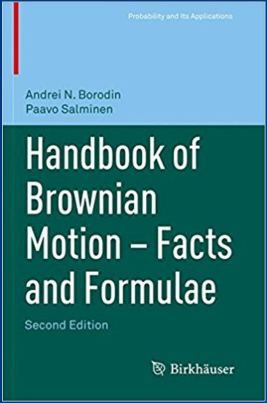
- Reflected Brownian Motion on $\mathbb{R}_+$: W_t =distance to 0 at time t.
- Hitting times: $H_z := \min_t \{W_t = z\}$.
- $$\mathbb{E}_x\left[e^{-\alpha H_z}\right] = \begin{cases} \frac{\cosh(xw)}{\cosh(zw)}, & 0 \le x \le z; \\ e^{-(x-z)w}, & z \le x. \end{cases}$$

1. $w = \sqrt{2\alpha}$;



Hitting Time

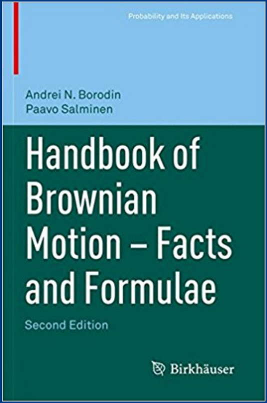
- Reflected Brownian Motion on $\mathbb{R}_+$: W_t =distance to 0 at time t.
- Hitting times: $H_z := \min_t \{W_t = z\}$.
- $$\mathbb{E}_x\left[e^{-\alpha H_z}\right] = \begin{cases} \frac{\cosh(xw)}{\cosh(zw)}, & 0 \le x \le z; \\ e^{-(x-z)w}, & z \le x. \end{cases}$$



1. $w = \sqrt{2\alpha}$;
2. $\mathbb{E}_x$ means it starts with point x (instead of 0);

Hitting Time

- Reflected Brownian Motion on $\mathbb{R}_+$:
  W_t =distance to 0 at time t.
- Hitting times: $H_z := \min_t \{W_t = z\}$.
- $$\mathbb{E}_x\left[e^{-\alpha H_z}\right] = \begin{cases} \frac{\cosh(xw)}{\cosh(zw)}, & 0 \le x \le z; \\ e^{-(x-z)w}, & z \le x. \end{cases}$$



1. $w = \sqrt{2\alpha}$;
2. $\mathbb{E}_x$ means it starts with point x (instead of 0);
3. $$\mathbb{E}\left[e^{s\left(iL-\frac{1}{2}\right)}\right] = \int_{\mathbb{R}} \frac{e^{s\left(it-\frac{1}{2}\right)}}{\cosh(\pi t)}\mathrm{d}t = \frac{e^{-\frac{s}{2}+sx}}{\cosh\left(\frac{s}{2}\right)}e^{sx}.$$
   $$\frac{2}{1+e^s}e^{sx} = \sum_{n=0}^{\infty} E_n(x)\frac{s^n}{n!}$$

Christophe's Idea

Consider a linear Brownian motion W_t starting from 0, with the hitting time T by W_t of level $z = 1$. Define another independent Brownian motion $\omega_t \sim \operatorname{sech}(x)$. Let

$$T_1 < T_2 < \cdots < T_l = T, \quad T_j = \min_s \left\{ W_t = \frac{j}{N} \right\}.$$

This defines a random walk with

$$p_\ell^{(N)} = \mathbb{P}\{W_t \text{ reach the sink in } \ell \text{ steps}\}.$$

Now write

$$T = (T - T_{\ell-1}) + (T_{\ell-1} - T_{\ell-2}) + \cdots + (T_1 - 0)$$

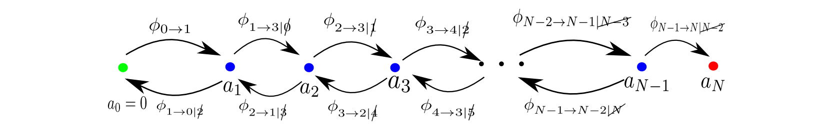
and

$$\omega_T \sim \omega_{T-T_{\ell-1}} + \omega_{T_{\ell-1}-T_{\ell-2}} + \cdots + \omega_{T_1-0},$$

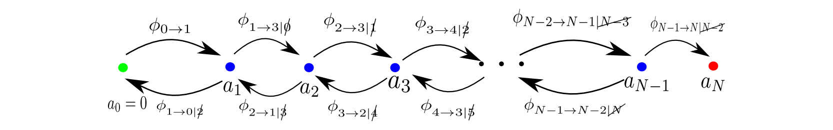
each term $\sim \operatorname{sech}(x)$. This corresponds Klebanov's random sum decomposition

$$Z_N = \frac{1}{N} \sum_{j=1}^{\nu_N} L_j \sim L.$$

My Goal

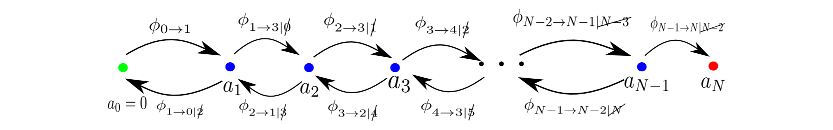

$\phi_{0\to1}$
$\phi_{1\to3|0}$
$\phi_{2\to3|1}$
$\phi_{3\to4|2}$
$\phi_{N-2\to N-1|N-3}$
$\phi_{N-1\to N|N-2}$
$a_0 = 0$
a_1
a_2
a_3
a_{N-1}
a_N
$\phi_{1\to0|2}$
$\phi_{2\to1|3}$
$\phi_{3\to2|4}$
$\phi_{4\to3|5}$
$\phi_{N-1\to N-2|N}$

My Goal



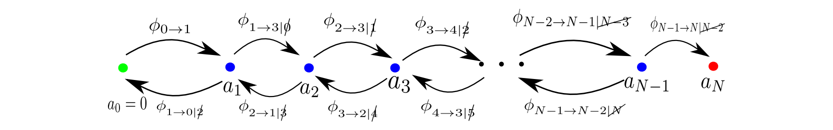
- A reflected Brownian motion model with N equally distributed sites; $a_j = j$ or j/N;

My Goal



- A reflected Brownian motion model with N equally distributed sites; $a_j = j$ or j/N;
- (modified) hitting times with "step" from one site to a nearby site $\sim \operatorname{sech}(x)$;

My Goal



- A reflected Brownian motion model with N equally distributed sites; $a_j = j$ or j/N;
- (modified) hitting times with "step" from one site to a nearby site $\sim \operatorname{sech}(x)$;

$$\mathbb{E}_x\left[e^{-\alpha H_z}\right] = \begin{cases} \frac{\cosh(xw)}{\cosh(zw)}, & 0 \leq x \leq z; \\ e^{-(x-z)w}, & z \leq x. \end{cases}$$

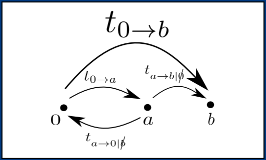
1-dim, 1-loop

With $p \leq q \leq r$, $w = \sqrt{2\alpha}$

$$\phi_{p \to q} := \mathbb{E}_p \left[e^{-\alpha H_q} \right] = \frac{\cosh(pw)}{\cosh(qw)},$$

$$\phi_{q \to p | \not{r}} := \mathbb{E}_q \left[e^{-\alpha H_p} | W_t < r \right] = \frac{\sinh((r-q)\,w)}{\sinh((r-p)\,w)},$$

$$\phi_{q \to r | \not{p}} := \mathbb{E}_q \left[e^{-\alpha H_r} | W_t > p \right] = \frac{\sinh((q-p)\,w)}{\sinh((r-p)\,w)},$$



1-dim, 1-loop

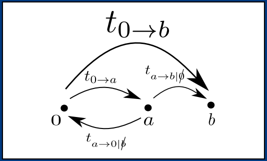
With $p \le q \le r$, $w = \sqrt{2\alpha}$

$$\phi_{p\to q} := \mathbb{E}_p\left[e^{-\alpha H_q}\right] = \frac{\cosh(pw)}{\cosh(qw)},$$

$$\phi_{q\to p|\not r} := \mathbb{E}_q\left[e^{-\alpha H_p}|W_t < r\right] = \frac{\sinh((r-q)\,w)}{\sinh((r-p)\,w)},$$

$$\phi_{q\to r|\not p} := \mathbb{E}_q\left[e^{-\alpha H_r}|W_t > p\right] = \frac{\sinh((q-p)\,w)}{\sinh((r-p)\,w)},$$

- The hitting time $t_{0\to b}$ can be decomposed as



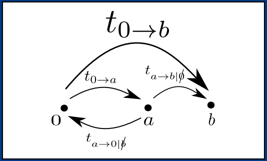
1-dim, 1-loop

With $p \leq q \leq r$, $w = \sqrt{2\alpha}$

$$\phi_{p\to q} := \mathbb{E}_p\left[e^{-\alpha H_q}\right] = \frac{\cosh(pw)}{\cosh(qw)},$$

$$\phi_{q\to p|\not r} := \mathbb{E}_q\left[e^{-\alpha H_p}|W_t < r\right] = \frac{\sinh((r-q)\,w)}{\sinh((r-p)\,w)},$$

$$\phi_{q\to r|\not p} := \mathbb{E}_q\left[e^{-\alpha H_r}|W_t > p\right] = \frac{\sinh((q-p)\,w)}{\sinh((r-p)\,w)},$$



- The hitting time $t_{0\to b}$ can be decomposed as

$$t_{0\to b} = \underbrace{\left(t_{0\to a} + t_{a\to 0|\not b}\right) + \cdots + \left(t_{0\to a} + t_{a\to 0|\not b}\right)}_{\ell \text{ copies}} + t_{0\to a} + t_{a\to b|\not 0}$$

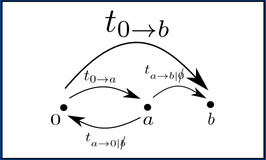
1-dim, 1-loop

With $p \le q \le r$, $w = \sqrt{2\alpha}$

$$\phi_{p\to q} := \mathbb{E}_p\left[e^{-\alpha H_q}\right] = \frac{\cosh(pw)}{\cosh(qw)},$$

$$\phi_{q\to p|\not r} := \mathbb{E}_q\left[e^{-\alpha H_p} | W_t < r\right] = \frac{\sinh((r-q)w)}{\sinh((r-p)w)},$$

$$\phi_{q\to r|\not p} := \mathbb{E}_q\left[e^{-\alpha H_r} | W_t > p\right] = \frac{\sinh((q-p)w)}{\sinh((r-p)w)},$$



- The hitting time $t_{0\to b}$ can be decomposed as

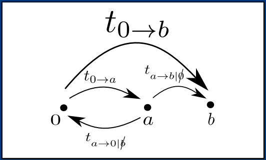
$$t_{0\to b} = \underbrace{\left(t_{0\to a} + t_{a\to 0|\not b}\right) + \cdots + \left(t_{0\to a} + t_{a\to 0|\not b}\right)}_{\ell \text{ copies}} + t_{0\to a} + t_{a\to b|\not 0}$$

- Generating functions:

$$\phi_{0\to b} = \phi_{0\to a}\phi_{a\to b|\not 0}\sum_{\ell=0}^{\infty}\left(\phi_{0\to a}\phi_{a\to 0|\not b}\right)^{\ell}$$

1-dim, 1-loop

With $p \le q \le r$, $w = \sqrt{2\alpha}$



$$\phi_{p\to q} := \mathbb{E}_p\left[e^{-\alpha H_q}\right] = \frac{\cosh(pw)}{\cosh(qw)},$$

$$\phi_{q\to p|\not r} := \mathbb{E}_q\left[e^{-\alpha H_p}|W_t < r\right] = \frac{\sinh((r-q)w)}{\sinh((r-p)w)},$$

$$\phi_{q\to r|\not p} := \mathbb{E}_q\left[e^{-\alpha H_r}|W_t > p\right] = \frac{\sinh((q-p)w)}{\sinh((r-p)w)},$$

- The hitting time $t_{0\to b}$ can be decomposed as

$$t_{0\to b} = \underbrace{\left(t_{0\to a} + t_{a\to 0|\not b}\right) + \cdots + \left(t_{0\to a} + t_{a\to 0|\not b}\right)}_{\ell \text{ copies}} + t_{0\to a} + t_{a\to b|\not 0}$$

- Generating functions:

$$\phi_{0\to b} = \phi_{0\to a}\phi_{a\to b|\not 0}\sum_{\ell=0}^{\infty}\left(\phi_{0\to a}\phi_{a\to 0|\not b}\right)^{\ell}$$

$$\begin{aligned}
\operatorname{sech}(bw) &= \operatorname{sech}(aw)\cdot\frac{\sinh(aw)}{\sinh(bw)}\sum_{\ell=0}^{\infty}\left[\operatorname{sech}(aw)\cdot\frac{\sinh((b-a)w)}{\sinh(bw)}\right]^{\ell} \\
&= \operatorname{sech}(aw)\cdot\frac{\sinh(aw)}{\sinh(bw)}\cdot\frac{1}{1-\operatorname{sech}(aw)\cdot\frac{\sinh((b-a)w)}{\sinh(bw)}}
\end{aligned}$$

1-dim, 1-loop

Prop. (**LJ** and C. Vignat, 2018)

$$E_n\left(\frac{x}{2b}+\frac{3}{2}-2\frac{a}{b}\right)-E_n\left(\frac{x}{b}+\frac{1}{2}\right)=\frac{(n+1)\left(1-2\frac{a}{b}\right)2^n a^n}{b^n}\sum_{\ell=0}^{\infty}\frac{\frac{a}{b}\left(1-\frac{a}{b}\right)^{\ell}}{}B_n^{(\ell+1)}\left(\frac{x+b}{4a}+\frac{\ell}{2}\right).$$

1-dim, 1-loop

Prop. (**LJ** and C. Vignat, 2018)

$$E_n\left(\frac{x}{2b}+\frac{3}{2}-2\frac{a}{b}\right)-E_n\left(\frac{x}{b}+\frac{1}{2}\right)=\frac{(n+1)\left(1-2\frac{a}{b}\right)2^n a^n}{b^n}\sum_{\ell=0}^{\infty}\underline{\frac{a}{b}\left(1-\frac{a}{b}\right)^{\ell}}B_n^{(\ell+1)}\left(\frac{x+b}{4a}+\frac{\ell}{2}\right).$$

- $\frac{a}{b}\left(1-\frac{a}{b}\right)^{\ell}$ are the probability weights of a geometric distribution with parameter a/b.

1-dim, 1-loop

Prop. (**LJ** and C. Vignat, 2018)

$$E_n\left(\frac{x}{2b}+\frac{3}{2}-2\frac{a}{b}\right)-E_n\left(\frac{x}{b}+\frac{1}{2}\right)=\frac{(n+1)\left(1-2\frac{a}{b}\right)2^n a^n}{b^n}\sum_{\ell=0}^{\infty}\underline{\frac{a}{b}\left(1-\frac{a}{b}\right)^{\ell}}B_n^{(\ell+1)}\left(\frac{x+b}{4a}+\frac{\ell}{2}\right).$$

- $\frac{a}{b}\left(1-\frac{a}{b}\right)^{\ell}$ are the probability weights of a geometric distribution with parameter a/b.

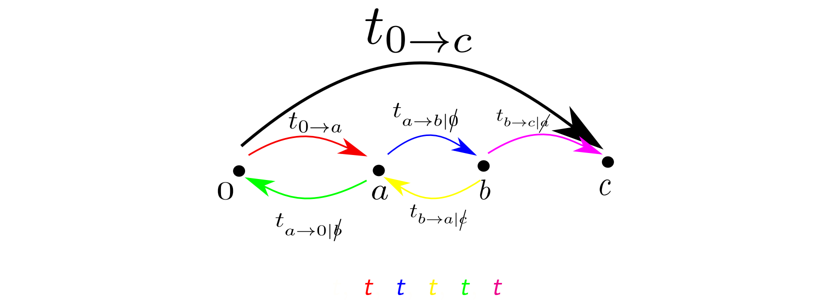
Definition

The **Bernoulli numbers** B_n, **Bernoulli polynomials** $B_n(x)$, and **Bernoulli polynomials of order** p, $B_n^{(p)}(x)$, are defined via their (exponential) generating functions

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!},\quad \frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\quad \left(\frac{t}{e^t-1}\right)^p e^{xt}=\sum_{n=0}^{\infty}B_n^{(p)}(x)\frac{t^n}{n!}.$$

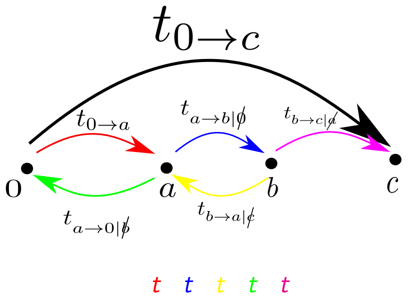
1-dim, 2-loops

How about 2-loops?



1-dim, 2-loops

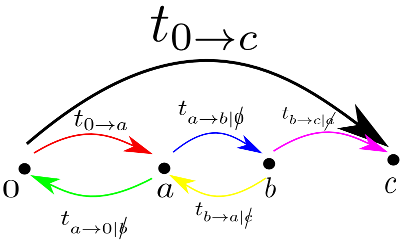
How about 2-loops?



$t =$

1-dim, 2-loops

How about 2-loops?

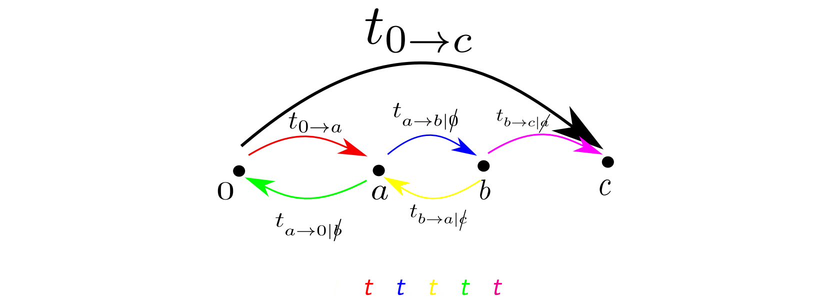


t t t t t

$t = t$

1-dim, 2-loops

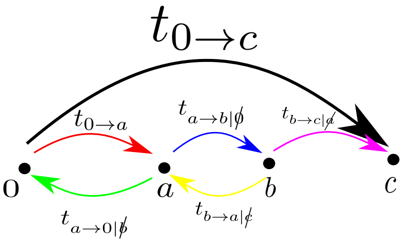
How about 2-loops?



$t = t + t$

1-dim, 2-loops

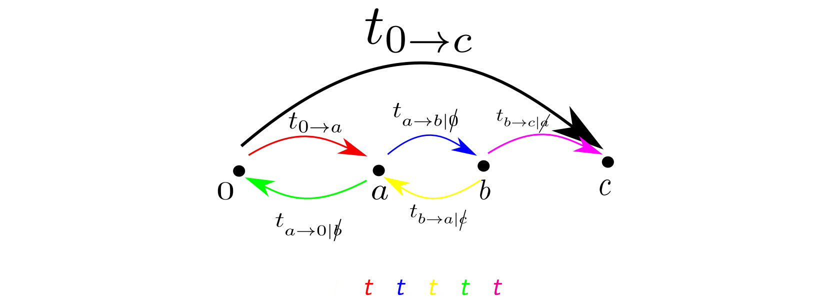
How about 2-loops?



t t t t t

$t \quad = \quad t + t + t$

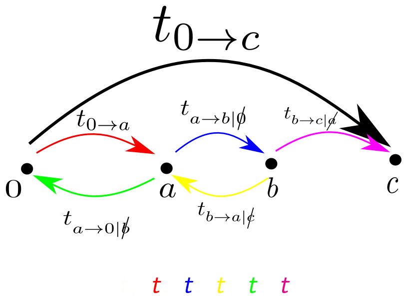
How about 2-loops?



$$t \;=\; t + t + t + t$$

1-dim, 2-loops

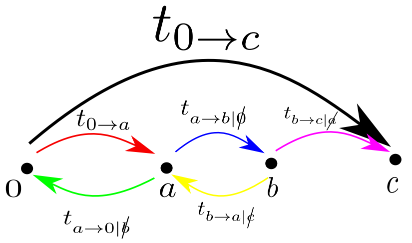
How about 2-loops?



$$t \quad = \quad t + t + t + t + t$$

1-dim, 2-loops

How about 2-loops?

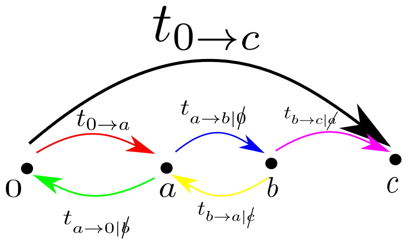


$t \quad t \quad t \quad t \quad t \quad t$

$$t \quad = \quad t + t + t + t + t + t$$

1-dim, 2-loops

How about 2-loops?

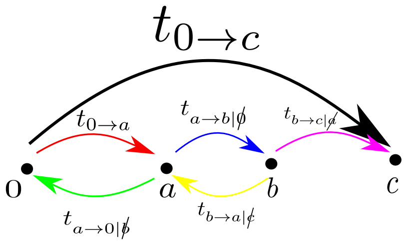


$t \quad t \quad t \quad t \quad t \quad t$

$$t \;=\; t + t + t + t + t + t + \cdots$$

1-dim, 2-loops

How about 2-loops?

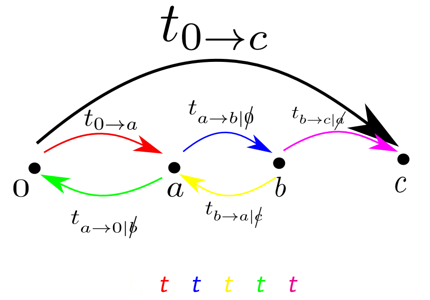


$t \quad t \quad t \quad t \quad t$

$$t \quad = \quad t + t + t + t + t + t + \cdots + t + t$$

1-dim, 2-loops

How about 2-loops?

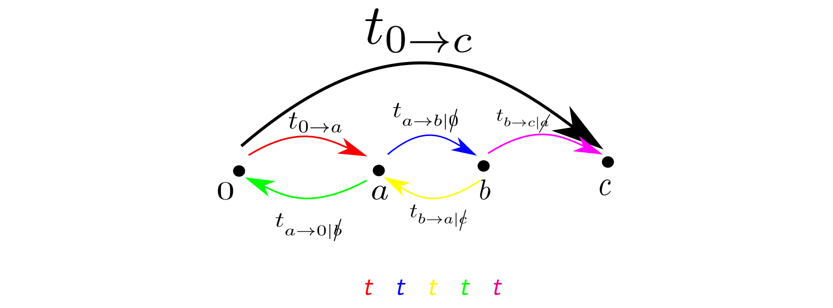


$$t = t + t + t + t + t + t + \cdots + t + t$$

$$= t + t + t + \underbrace{(t + t) + \cdots + (t + t)}_{k \text{ loops}} + \underbrace{(t + t) + \cdots + (t + t)}_{\ell \text{ loops}}$$

1-dim, 2-loops

How about 2-loops?

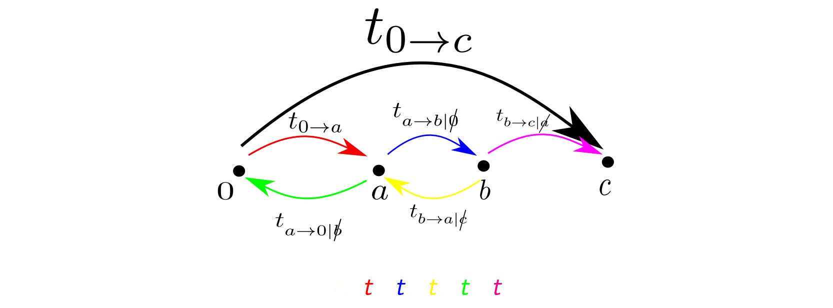


$$
\begin{aligned}
t &= t + t + t + t + t + t + \cdots + t + t \\
&= t + t + t + \underbrace{(t + t) + \cdots + (t + t)}_{k \text{ loops}} + \underbrace{(t + t) + \cdots + (t + t)}_{\ell \text{ loops}}
\end{aligned}
$$

We can generalize it to n-loop model.

1-dim, 2-loops

How about 2-loops?

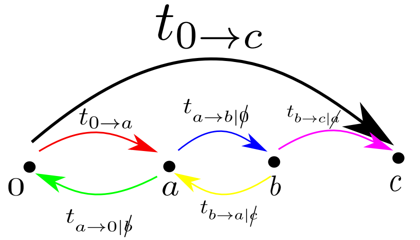


$$
\begin{aligned}
t &= t + t + t + t + t + t + \cdots + t + t \\
&= t + t + t + \underbrace{(t + t) + \cdots + (t + t)}_{k \text{ loops}} + \underbrace{(t + t) + \cdots + (t + t)}_{\ell \text{ loops}}
\end{aligned}
$$

We can generalize it to n-loop model.

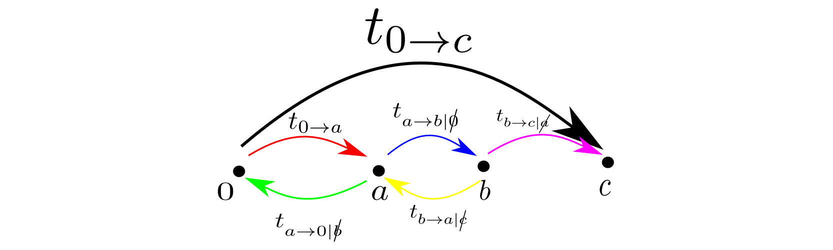
Unfortunately, this is WRONG.....

1-dim, 2-loops



$$\phi = \phi \cdot \phi \cdot \phi \cdot \left[\sum_{k=0}^{\infty} (\phi\phi)^k\right]\left[\sum_{\ell=0}^{\infty} (\phi\phi)^\ell\right] = \frac{\phi \cdot \phi \cdot \phi}{(1-\phi\phi)\,(1-\phi\phi)}$$

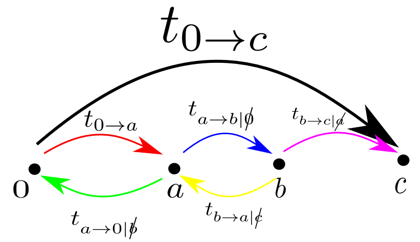
1-dim, 2-loops



$$\phi = \phi \cdot \phi \cdot \phi \cdot \left[\sum_{k=0}^{\infty} (\phi\phi)^k \right] \left[\sum_{\ell=0}^{\infty} (\phi\phi)^\ell \right] = \frac{\phi \cdot \phi \cdot \phi}{(1 - \phi\phi)\,(1 - \phi\phi)}$$

Let $a = 1$, $b = 2$ and $c = 3$.

1-dim, 2-loops



$$\phi = \phi \cdot \phi \cdot \phi \cdot \left[\sum_{k=0}^{\infty} (\phi\phi)^k\right] \left[\sum_{\ell=0}^{\infty} (\phi\phi)^\ell\right] = \frac{\phi \cdot \phi \cdot \phi}{(1-\phi\phi)(1-\phi\phi)}$$

Let $a = 1,\ b = 2$ and $c = 3$.

$$\text{LHS} = \phi = \phi_{0\to 3} = \operatorname{sech}(3w) = \frac{1}{\cosh(3w)}$$

$$\text{RHS} = \frac{\frac{1}{\cosh(w)} \cdot \frac{\sinh(w)}{\sinh(2w)} \cdot \frac{\sinh(w)}{\sinh(2w)}}{\left(1 - \frac{1}{\cosh(w)} \frac{\sinh(w)}{\sinh(2w)}\right)\left(1 - \frac{\sinh(w)}{\sinh(2w)} \frac{\sinh(w)}{\sinh(2w)}\right)} = \frac{\frac{1}{4\cosh^3 w}}{\left(1 - \frac{1}{2\cosh^2 w}\right)\left(1 - \frac{1}{4\cosh^2 w}\right)}$$

$$= \frac{2\cosh w}{(2\cosh^2 w - 1)(4\cosh^2 w - 1)} \neq \frac{1}{\cosh(3w)} = \frac{1}{4\cosh^3 w - 3\cosh w}$$

Two-loops

$$I := \phi_{a\to b|\not{A}}\phi_{b\to a|\not{c}}, \quad II := \phi_{b\to c|\not{a}}\phi_{c\to b|\not{B}}$$

- k loops of I followed by l loops of II, with $k, l = 0, 1, \dots,$ which gives

$$\sum_{k,l} I^k II^l = \frac{1}{1-I}\cdot\frac{1}{1-II};$$

- k_1 loops of I followed by l_1 loops of II, then followed by k_2 loops of I and finally followed by l_2 loops of II, with k_1, l_2 nonnegative and k_2, l_1 positive, which gives

$$\sum_{k_1,l_2=0,k_2,l_1=1}^{\infty} I^{k_1} II^{l_1} I^{k_2} II^{l_2} = \frac{I\cdot II}{(1-I)^2(1-II)^2};$$

- the general term will be k_1 loops of $I \to l_1$ loops of $II \to \cdots \to k_n$ loops of $I \to l_n$ loops of II, with k_1, l_n nonnegative and the rest indices positive, which gives

$$\frac{(I\cdot II)^{n-1}}{(1-I)^n(1-II)^n}.$$

Therefore, loops I and II contribute as

$$\sum_{n=1}^{\infty}\frac{(I\cdot II)^{n-1}}{(1-I)^n(1-II)^n} = \frac{1}{1-(I+II)} = \sum_{k=0}^{\infty}(I+II)^k.$$

Two Loops

Prop. (LJ. and C. Vignat, 2018)

For any positive integer n,

$$E_n\left(\frac{x}{6}\right) = \sum_{k=0}^{\infty} \frac{3^{k-n}}{4^{k+1}} E_n^{(2k+3)}\left(\frac{x}{2}+k\right).$$

Two Loops

Prop. (LJ. and C. Vignat, 2018)

For any positive integer n,

$$E_n\left(\frac{x}{6}\right) = \sum_{k=0}^{\infty} \frac{3^{k-n}}{4^{k+1}} E_n^{(2k+3)}\left(\frac{x}{2}+k\right).$$

In general

$$(x+2c\mathcal{E}+c)^n = \sum_{k=0}^{\infty}\sum_{\ell=0}^{k} q_{k,\ell}\Big[x+2(b-a)\mathcal{B}+2(c-b)\mathcal{B}'+a\mathcal{E}^{(\ell)}+2(b-a)\mathcal{U}^{(l)}$$
$$+2a\mathcal{U}'^{(k-\ell)}+2(b-a)\mathcal{B}'^{(k-\ell)}+q'_{k,\ell}\Big]^n,$$

$$q_{k,\ell} := \binom{k}{\ell}\frac{(b-a)^{\ell+1}\,a^{k-\ell+1}\,(c-b)^{k-\ell}}{b^{k+1}\,(c-a)^{k-\ell+1}} \qquad q'_{k,\ell} = c+(2k-2\ell)\,b+(3\ell-k+1)\,a,$$

where

$$\left(\mathcal{E}^{(p)}+x\right)^n = E_n^{(p)}(x),\quad \left(\mathcal{B}^{(p)}+x\right)^n = B_n^{(p)}(x),\quad \mathcal{U}^n = \frac{1}{n+1},\quad \mathcal{U}^{(p)} = \mathcal{U}_1+\cdots+\mathcal{U}_p.$$

n loops?

Consider consecutive loops l_1, l_2, ..., l_n, it seems like the contribution is

$$\sum_{k=0}^{\infty} \left(\sum_{\ell=1}^{n} l_\ell \right)^k = \frac{1}{1-(l_1+\cdots+l_n)}. \qquad (*)$$

n loops?

Consider consecutive loops l_1, l_2, ..., l_n, it seems like the contribution is

$$\sum_{k=0}^{\infty}\left(\sum_{\ell=1}^{n} l_\ell\right)^k = \frac{1}{1-(l_1+\cdots+l_n)}. \qquad (*)$$

- It feels right.

n loops?

Consider consecutive loops l_1, l_2, ..., l_n, it seems like the contribution is

$$\sum_{k=0}^{\infty}\left(\sum_{\ell=1}^{n} l_\ell\right)^k = \frac{1}{1-(l_1+\cdots+l_n)}. \qquad (*)$$

- It feels right.
- I can "prove" it by induction.

n loops?

Consider consecutive loops l_1, l_2, ..., l_n, it seems like the contribution is

$$\sum_{k=0}^{\infty}\left(\sum_{\ell=1}^{n} l_\ell\right)^k = \frac{1}{1-(l_1+\cdots+l_n)}. \qquad (*)$$

- It feels right.
- I can "prove" it by induction.
- In general sites $0, 1, \ldots, N$:

n loops?

Consider consecutive loops l_1, l_2, ..., l_n, it seems like the contribution is

$$\sum_{k=0}^{\infty}\left(\sum_{\ell=1}^{n} l_\ell\right)^k = \frac{1}{1-(l_1+\cdots+l_n)}. \qquad (*)$$

- It feels right.
- I can "prove" it by induction.
- In general sites $0, 1, \ldots, N$:

$$\frac{1}{\cosh(Nw)} \overset{??}{=} \frac{\frac{1}{\cosh w}\cdot\left(\frac{\sinh w}{\sinh(2w)}\right)^N}{1-\left(\frac{1}{\cosh(w)}\frac{\sinh(w)}{\sinh(2w)}+(N-1)\frac{\sinh(w)}{\sinh(2w)}\frac{\sinh(w)}{\sinh(2w)}\right)}$$

$$= \frac{\frac{1}{2^N\cosh^{N+1} w}}{1-\frac{N+3}{4}\cosh^N w}.$$

This shows $(*)$ is not correct.

$$\frac{1}{\cosh(Nw)} = \frac{1}{\cos(Niw)} = \frac{1}{T_N(\cos(iw))} = \frac{1}{T_N(\cosh w)}.$$

General Formula

Thm. (LJ, I. Simonelli, and H. Yue, 2021-2022)

$$\phi_{0\to a_n} = \phi_{0\to a_{\mathbf{1}}}\phi_{a_{\mathbf{1}}\to a_{\mathbf{2}}|\emptyset}\cdots\phi_{a_{n-\mathbf{1}}\to a_n|\cancel{a_{n-\mathbf{2}}}}\cdot\frac{1}{1-P(L_1,\ldots,L_n)},$$

General Formula

Thm. (**LJ**, I. Simonelli, and H. Yue, 2021-2022)

$$\phi_{0\to a_n} = \phi_{0\to a_1}\phi_{a_1\to a_2|\emptyset}\cdots\phi_{a_{n-1}\to a_n|\cancel{a_{n-2}}}\cdot\frac{1}{1-P(L_1,\ldots,L_n)},$$

where

$$L_j = \phi_{a_{j-1}\to a_j|\cancel{a_{j-2}}}\phi_{a_j\to a_{j-1}|\cancel{a_{j+1}}}$$

$$P\left(L_1,\ldots,L_n\right) = \sum_{*}(-1)^{\ell+1}L_{j_1}\cdots L_{j_\ell},$$

General Formula

Thm. (LJ, I. Simonelli, and H. Yue, 2021-2022)

$$\phi_{0\to a_n} = \phi_{0\to a_1}\phi_{a_1\to a_2|\cancel{\emptyset}}\cdots\phi_{a_{n-1}\to a_n|\cancel{a_{n-2}}}\cdot\frac{1}{1-P(L_1,\ldots,L_n)},$$

where

$$L_j = \phi_{a_{j-1}\to a_j|\cancel{a_{j-2}}}\phi_{a_j\to a_{j-1}|\cancel{a_{j+1}}}$$
$$P\left(L_1,\ldots,L_n\right) = \sum_{*}(-1)^{\ell+1}L_{j_1}\cdots L_{j_\ell},$$

for the condition $*$ given by

- $\ell = 1, 2, \ldots, n$;
- $j_1 < j_2 - 1$, $j_2 < j_3 - 1$, $\ldots$, $j_{\ell-1} < j_\ell - 1$.

Examples

- $n = 2$: $P = L_1 + L_2$;

Examples

- $n = 2$: $P = L_1 + L_2$;
- $n = 3$: $P = L_1 + L_2 + L_3 - L_1 \cdot L_3$

Examples

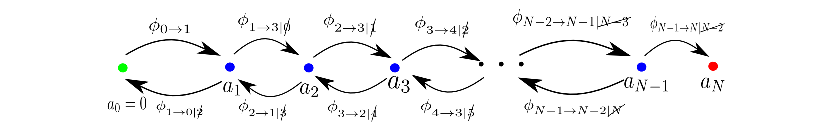
- $n = 2$: $P = L_1 + L_2$;
- $n = 3$: $P = L_1 + L_2 + L_3 - L_1 \cdot L_3$
- $n = 4$: $P = L_1 + \cdots + L_4 - L_1 L_3 - L_1 L_4 - L_2 L_4$

Examples

- $n = 2$: $P = L_1 + L_2$;
- $n = 3$: $P = L_1 + L_2 + L_3 - L_1 \cdot L_3$
- $n = 4$: $P = L_1 + \cdots + L_4 - L_1 L_3 - L_1 L_4 - L_2 L_4$
- $n = 5$: $P = L_1 + \cdots + L_5 - L_1 L_3 - \cdots - L_3 L_5 + L_1 L_3 L_5$.

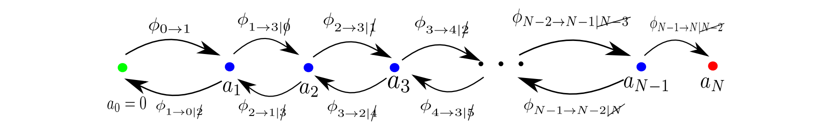
Examples

- $n = 2$: $P = L_1 + L_2$;
- $n = 3$: $P = L_1 + L_2 + L_3 - L_1 \cdot L_3$
- $n = 4$: $P = L_1 + \cdots + L_4 - L_1 L_3 - L_1 L_4 - L_2 L_4$
- $n = 5$: $P = L_1 + \cdots + L_5 - L_1 L_3 - \cdots - L_3 L_5 + L_1 L_3 L_5$.



Examples

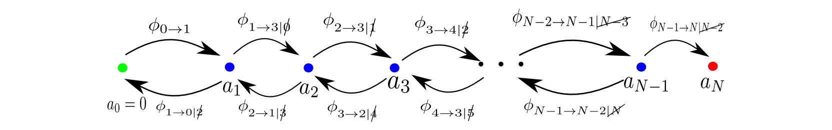
- $n=2$: $P = L_1 + L_2$;
- $n=3$: $P = L_1 + L_2 + L_3 - L_1 \cdot L_3$
- $n=4$: $P = L_1 + \cdots + L_4 - L_1 L_3 - L_1 L_4 - L_2 L_4$
- $n=5$: $P = L_1 + \cdots + L_5 - L_1 L_3 - \cdots - L_3 L_5 + L_1 L_3 L_5$.



1. Induction.

Examples

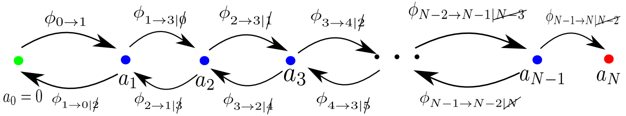
- $n = 2$: $P = L_1 + L_2$;
- $n = 3$: $P = L_1 + L_2 + L_3 - L_1 \cdot L_3$
- $n = 4$: $P = L_1 + \cdots + L_4 - L_1 L_3 - L_1 L_4 - L_2 L_4$
- $n = 5$: $P = L_1 + \cdots + L_5 - L_1 L_3 - \cdots - L_3 L_5 + L_1 L_3 L_5$.



1. Induction. The tricky part is, if we "glue" the first two loops together; or ignore site a_1, $\phi_{2\to 3|\not 1}$ should be replaced by $\phi_{2\to 3|\emptyset}$.

Examples

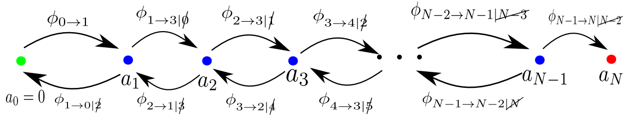
- $n = 2$: $P = L_1 + L_2$;
- $n = 3$: $P = L_1 + L_2 + L_3 - L_1 \cdot L_3$
- $n = 4$: $P = L_1 + \cdots + L_4 - L_1 L_3 - L_1 L_4 - L_2 L_4$
- $n = 5$: $P = L_1 + \cdots + L_5 - L_1 L_3 - \cdots - L_3 L_5 + L_1 L_3 L_5$.



1. Induction. The tricky part is, if we "glue" the first two loops together; or ignore site a_1, $\phi_{2\to3|\not{1}}$ should be replaced by $\phi_{2\to3|\not{0}}$.
2. Inclusion-exclusion principle

Examples

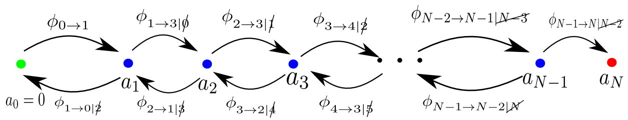
- $n = 2$: $P = L_1 + L_2$;
- $n = 3$: $P = L_1 + L_2 + L_3 - L_1 \cdot L_3$
- $n = 4$: $P = L_1 + \cdots + L_4 - L_1 L_3 - L_1 L_4 - L_2 L_4$
- $n = 5$: $P = L_1 + \cdots + L_5 - L_1 L_3 - \cdots - L_3 L_5 + L_1 L_3 L_5$.



1. Induction. The tricky part is, if we "glue" the first two loops together; or ignore site a_1, $\phi_{2\to3|\not 1}$ should be replaced by $\phi_{2\to3|\not 0}$.
2. Inclusion-exclusion principle

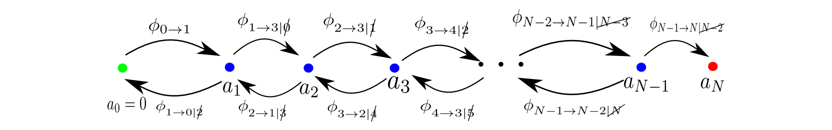
Inclusion-exclusion principle

- $n = 2$: $P = L_1 + L_2$;
- $n = 3$: $P = L_1 + L_2 + L_3 - L_3 \cdot L_1$
- $n = 4$: $P = L_1 + \cdots + L_4 - L_3 L_1 - L_4 L_1 - L_4 L_2$
- $n = 5$: $P = L_1 + \cdots + L_5 - L_3 L_1 - \cdots - L_5 L_3 + L_5 L_3 L_1$.



Inclusion-exclusion principle

- $n = 2$: $P = L_1 + L_2$;
- $n = 3$: $P = L_1 + L_2 + L_3 - L_3 \cdot L_1$
- $n = 4$: $P = L_1 + \cdots + L_4 - L_3 L_1 - L_4 L_1 - L_4 L_2$
- $n = 5$: $P = L_1 + \cdots + L_5 - L_3 L_1 - \cdots - L_5 L_3 + L_5 L_3 L_1$.



$$\frac{1}{1-(L_1+L_2+L_3-L_3\cdot L_1)} = \sum_{k=0}^{\infty} (L_1+L_2+L_3-L_3\cdot L_1)^k .$$

Results

Thm. (**LJ**, I. Simonelli, and H. Yue, 2021–2022)

By letting $a_j = j$, we have

$$E_n(x) = \frac{1}{4^n} \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \frac{2^n}{8^{\ell+1}} E_n^{(2k+2\ell)} (4x + k + \ell).$$

$$E_n(x) = \sum_{k=0}^{\infty} \frac{5^{k-n}}{4^{k+\ell+2}} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} E_n^{(2\ell+2k+5)} (5x + \ell + k).$$

$\ldots$

Generalization

- Bessel process in $\mathbb{R}^n$:

$$R_t^{(n)} := \sqrt{\left(W_t^{(1)}\right)^2 + \cdots + \left(W_t^{(n)}\right)^2}$$

- Moment generating functions for hitting times:
$H_z := \min_s \left\{ R_s^{(n)} = z \right\}$.

$$\mathbb{E}_x \left(e^{-\alpha H_z}; \sup_{0 \le s \le H_z} R_s^{(n)} < y \right) = \begin{cases} \frac{x^{-\nu} I_\nu(xw)}{z^{-\nu} I_\nu(zw)}, & 0 \le x \le z \le y; \\ \frac{S_\nu(yw,xw)}{S_\nu(yw,zw)}, & z \le x \le y, \end{cases}$$

- $n = 2 + 2\nu$ for $\nu \ge 0$

$$S_\nu(x,y) := (xy)^{-\nu} \left[I_\nu(x) K_\nu(y) - K_\nu(x) I_\nu(y) \right],$$

and

$$I_\nu(x) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!\Gamma(\ell+\nu+1)} \left(\frac{x}{2}\right)^{2\ell+\nu}, \quad K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\nu\pi)}.$$

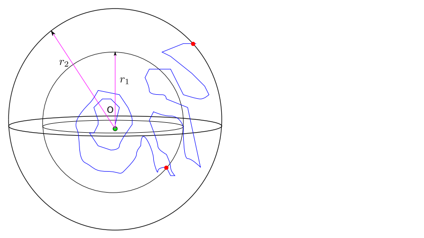
$n = 3 \Leftrightarrow \nu = 1/2$

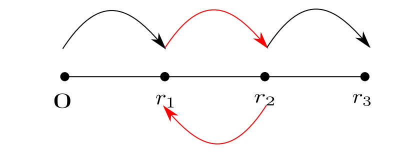
- $$I_{\frac{1}{2}}(x) = \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2m+\frac{1}{2}}}{m!\Gamma\left(m+\frac{3}{2}\right)} = \sqrt{\frac{2}{x\pi}}\sinh(x)$$

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t - 1}e^{tx} = \frac{te^{tx}e^{-\frac{t}{2}}}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} = \frac{te^{t\left(x-\frac{1}{2}\right)}}{2}\sinh\left(\frac{t}{2}\right)$$

$$K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}$$

$$\mathbb{E}_x\left(e^{-\alpha H_z};\ \sup_{0\le s\le H_z} R_s^{(3)} < y\right) = \begin{cases} \frac{z\sinh(xw)}{x\sinh(zw)}, & 0 \le x \le z \le y \\ \frac{z\sinh((y-x)w)}{x\sinh((y-z)w)}, & z \le x \le y \end{cases}$$





$n = 3 \Leftrightarrow \nu = 1/2, r_1 = 1, r_2 = 2, r_3 = 3$

Prop. (LJ. and C. Vignat, 2018)

$$\frac{3^{n+1}}{n+1}\left[B_{n+1}\left(\frac{x}{6}+\frac{5}{6}\right) - B_{n+1}\left(\frac{x}{6}+\frac{1}{2}\right)\right] = \sum_{k\geq 0}\frac{3}{4}\left(\frac{1}{4}\right)^k E_n^{(2k+2)}\left(\frac{x+3+2k}{2}\right).$$

$n = 3 \Leftrightarrow \nu = 1/2, r_1 = 1, r_2 = 2, r_3 = 3$

Prop. (LJ. and C. Vignat, 2018)

$$\frac{3^{n+1}}{n+1}\left[B_{n+1}\left(\frac{x}{6}+\frac{5}{6}\right)-B_{n+1}\left(\frac{x}{6}+\frac{1}{2}\right)\right]=\sum_{k\geq 0}\frac{3}{4}\left(\frac{1}{4}\right)^{k}E_{n}^{(2k+2)}\left(\frac{x+3+2k}{2}\right). \qquad \circledast$$

Corollary

1. *Take* $x = 0$, $n = 2m-1$ *in* ($\circledast$).

$$B_{2m}=\frac{m}{\left(1-2^{1-2m}\right)\left(3^{2m}-1\right)}\sum_{k\geq 0}\left(\frac{1}{4}\right)^{k}E_{2m-1}^{(2k+2)}\left(k+\frac{3}{2}\right).$$

$n = 3 \Leftrightarrow \nu = 1/2, r_1 = 1, r_2 = 2, r_3 = 3$

Prop. (LJ. and C. Vignat, 2018)

$$\frac{3^{n+1}}{n+1}\left[B_{n+1}\left(\frac{x}{6}+\frac{5}{6}\right) - B_{n+1}\left(\frac{x}{6}+\frac{1}{2}\right)\right] = \sum_{k\geq 0}\frac{3}{4}\left(\frac{1}{4}\right)^k E_n^{(2k+2)}\left(\frac{x+3+2k}{2}\right). \quad \circledast$$

Corollary

1. *Take* $x = 0,\ n = 2m-1$ *in* ($\circledast$).

$$B_{2m} = \frac{m}{\left(1-2^{1-2m}\right)\left(3^{2m}-1\right)}\sum_{k\geq 0}\left(\frac{1}{4}\right)^k E_{2m-1}^{(2k+2)}\left(k+\frac{3}{2}\right).$$

2. *Take* $n = 1$ *in* ($\circledast$).

$$\sum_{k\geq 0}\frac{3}{4}\left(\frac{1}{4}\right)^k\left(\frac{x+3+2k}{2}-k-1\right) = \sum_{k\geq 0}\frac{3}{4}\left(\frac{1}{4}\right)^k\left(\frac{x+1}{2}\right) = \frac{x+1}{2}.$$

$n = 3 \Leftrightarrow \nu = 1/2, r_1 = 1, r_2 = 2, r_3 = 3$

Prop. (LJ. and C. Vignat, 2018)

$$\frac{3^{n+1}}{n+1}\left[B_{n+1}\left(\frac{x}{6}+\frac{5}{6}\right) - B_{n+1}\left(\frac{x}{6}+\frac{1}{2}\right)\right] = \sum_{k\geq 0}\frac{3}{4}\left(\frac{1}{4}\right)^k E_n^{(2k+2)}\left(\frac{x+3+2k}{2}\right). \qquad \circledast$$

Corollary

1. *Take* $x = 0$, $n = 2m-1$ *in* ($\circledast$).

$$B_{2m} = \frac{m}{\left(1-2^{1-2m}\right)\left(3^{2m}-1\right)}\sum_{k\geq 0}\left(\frac{1}{4}\right)^k E_{2m-1}^{(2k+2)}\left(k+\frac{3}{2}\right).$$

2. *Take* $n = 1$ *in* ($\circledast$).

$$\sum_{k\geq 0}\frac{3}{4}\left(\frac{1}{4}\right)^k\left(\frac{x+3+2k}{2}-k-1\right) = \sum_{k\geq 0}\frac{3}{4}\left(\frac{1}{4}\right)^k\left(\frac{x+1}{2}\right) = \frac{x+1}{2}.$$

Prop. (**LJ** and C. Vignat, 2018)

For any positive integer n,

$$3^n B_n\left(\frac{x+4}{6}\right) = \sum_{k=0}^{\infty}\frac{1}{2^k}E_n^{(2k+2)}\left(\frac{x+2k+3}{2}\right).$$

Results

Thm. (LJ, I. Simonelli, and H. Yue, 2021–2022)

For any $m \in \mathbb{N}$and let $M = m-1$ and M' be the largest odd number less than or equal to M, then

$$B_{n+1}\left(\frac{2+x}{m+2}\right) - B_{n+1}\left(\frac{x}{m+2}\right)$$
$$= \frac{n+1}{(m+2)^n}\sum_{k=0}^{\infty}\frac{m^k}{2^{2k+m}}\sum_{n_1,\ldots,n_M=0}^{k}\binom{k}{n_1,\ldots,n_M}(-1)^{n_1+n_3+\cdots+n_{M'}}$$
$$\times \frac{\binom{m-1}{2}^{n_1}\binom{m-2}{3}^{n_2}\cdots\binom{m-M}{M}^{n_M}}{4^{n_1+2n_2+Mn_M}\, m^{n_1+\cdots+n_M}}$$
$$\times E_n^{(2k+2n_1+4n_2+\cdots+2Mn_M+m)}(k+n_1+2n_2+\cdots+Mn_M+x).$$

Example

$$B_{n+1}\left(\frac{x+2}{5}\right) - B_{n+1}\left(\frac{x}{5}\right) = \frac{n+1}{5^n}\sum_{k=0}^{\infty}\frac{3^k}{2^{2k+3}}\sum_{\ell=0}^{k}\binom{k}{\ell}\frac{(-1)^\ell}{12^\ell}E_n^{(2k+2\ell+3)}(k+\ell+x).$$

Example

$$B_{n+1}\left(\frac{x+2}{5}\right) - B_{n+1}\left(\frac{x}{5}\right) = \frac{n+1}{5^n}\sum_{k=0}^{\infty}\frac{3^k}{2^{2k+3}}\sum_{\ell=0}^{k}\binom{k}{\ell}\frac{(-1)^\ell}{12^\ell}E_n^{(2k+2\ell+3)}(k+\ell+x).$$

$$\sum_{n=0}^{\infty}\left(B_{n+1}\left(\frac{x+2}{5}\right) - B_{n+1}\left(\frac{x}{5}\right)\right)\frac{t^{n+1}}{(n+1)!}$$

Example

$$B_{n+1}\left(\frac{x+2}{5}\right)-B_{n+1}\left(\frac{x}{5}\right)=\frac{n+1}{5^n}\sum_{k=0}^{\infty}\frac{3^k}{2^{2k+3}}\sum_{\ell=0}^{k}\binom{k}{\ell}\frac{(-1)^\ell}{12^\ell}E_n^{(2k+2\ell+3)}(k+\ell+x).$$

$$\sum_{n=0}^{\infty}\left(B_{n+1}\left(\frac{x+2}{5}\right)-B_{n+1}\left(\frac{x}{5}\right)\right)\frac{t^{n+1}}{(n+1)!}$$

$$=\quad\frac{t\left(e^{\frac{2t}{5}}-1\right)}{e^t-1}e^{\frac{tx}{5}}$$

Example

$$B_{n+1}\left(\frac{x+2}{5}\right)-B_{n+1}\left(\frac{x}{5}\right)=\frac{n+1}{5^n}\sum_{k=0}^{\infty}\frac{3^k}{2^{2k+3}}\sum_{\ell=0}^{k}\binom{k}{\ell}\frac{(-1)^\ell}{12^\ell}E_n^{(2k+2\ell+3)}(k+\ell+x).$$

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(B_{n+1}\left(\frac{x+2}{5}\right)-B_{n+1}\left(\frac{x}{5}\right)\right)\frac{t^{n+1}}{(n+1)!}\\
=\;&\frac{t\left(e^{\frac{2t}{5}}-1\right)}{e^t-1}e^{\frac{tx}{5}}\\
=\;&t\sum_{n=0}^{\infty}\left(\frac{1}{5^n}\sum_{k=0}^{\infty}\frac{3^k}{2^{2k+3}}\sum_{\ell=0}^{k}\binom{k}{\ell}\frac{(-1)^\ell}{12^\ell}E_n^{(2k+2\ell+3)}(k+\ell+x)\right)\frac{t^n}{n!}.
\end{aligned}$$

Future Work

Future Work

- Uniqueness.

Future Work

- Uniqueness.

Prop. (LJ. and C. Vignat, 2018)

$$B_n\left(\frac{x+4}{6}\right) = \frac{1}{3^n}\sum_{k=0}^{\infty}\frac{E_n^{(2k+2)}\left(\frac{x+2k+3}{2}\right)}{2^k}$$

Future Work

- Uniqueness.

Prop. (LJ. and C. Vignat, 2018)

$$B_n\left(\frac{x+4}{6}\right) = \frac{1}{3^n}\sum_{k=0}^{\infty}\frac{E_n^{(2k+2)}\left(\frac{x+2k+3}{2}\right)}{2^k}$$

$$\frac{4w}{\sinh(4w)} = \frac{w}{\sinh w}\frac{2\sinh w}{\sinh(2w)}\frac{3\sinh w}{2\sinh(2w)}\frac{4\sinh w}{3\sinh(2w)}\sum_{k=0}^{\infty}\frac{\operatorname{sech}^{2k} w}{2^k}$$

$$= \frac{w}{\sinh w}\sum_{k=0}^{\infty}\frac{\operatorname{sech}^{2k+3} w}{2^{k+1}}$$

Future Work

- Uniqueness.

Prop. (LJ. and C. Vignat, 2018)

$$B_n\left(\frac{x+4}{6}\right)=\frac{1}{3^n}\sum_{k=0}^{\infty}\frac{E_n^{(2k+2)}\left(\frac{x+2k+3}{2}\right)}{2^k}$$

$$\frac{4w}{\sinh(4w)}=\frac{w}{\sinh w}\frac{2\sinh w}{\sinh(2w)}\frac{3\sinh w}{2\sinh(2w)}\frac{4\sinh w}{3\sinh(2w)}\sum_{k=0}^{\infty}\frac{\operatorname{sech}^{2k} w}{2^k}$$
$$=\frac{w}{\sinh w}\sum_{k=0}^{\infty}\frac{\operatorname{sech}^{2k+3} w}{2^{k+1}}$$

- General formula

2-loop, 3-dim

In general

$$(x+2c\mathcal{E}+c)^n = \sum_{k=0}^{\infty}\sum_{\ell=0}^{k} q_{k,\ell}\Big[x+2\,(b-a)\,\mathcal{B}+2\,(c-b)\,\mathcal{B}'+a\mathcal{E}^{(\ell)}+2\,(b-a)\,\mathcal{U}^{(l)}$$
$$+2a\mathcal{U}'^{(k-\ell)}+2\,(b-a)\,\mathcal{B}'^{(k-\ell)}+q'_{k,\ell}\Big]^n,$$

$$q_{k,\ell} := \binom{k}{\ell}\frac{(b-a)^{\ell+1}\,a^{k-\ell+1}\,(c-b)^{k-\ell}}{b^{k+1}\,(c-a)^{k-\ell+1}} \qquad q'_{k,\ell} = c+(2k-2\ell)\,b+(3\ell-k+1)\,a,$$

where

$$\left(\mathcal{E}^{(p)}+x\right)^n = E_n^{(p)}(x),\quad \left(\mathcal{B}^{(p)}+x\right)^n = B_n^{(p)}(x),\quad \mathcal{U}^n = \frac{1}{n+1},\quad \mathcal{U}^{(p)} = \mathcal{U}_1+\cdots+\mathcal{U}_p.$$

2-loop, 3-dim

In general

$$(x + 2c\mathcal{E} + c)^n = \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} q_{k,\ell} \Big[x + 2(b-a)\mathcal{B} + 2(c-b)\mathcal{B}' + a\mathcal{E}^{(\ell)} + 2(b-a)\mathcal{U}^{(l)} + 2a\mathcal{U}'^{(k-\ell)} + 2(b-a)\mathcal{B}'^{(k-\ell)} + q'_{k,\ell} \Big]^n,$$

$$q_{k,\ell} := \binom{k}{\ell} \frac{(b-a)^{\ell+1}\, a^{k-\ell+1}\, (c-b)^{k-\ell}}{b^{k+1}\,(c-a)^{k-\ell+1}} \qquad q'_{k,\ell} = c + (2k-2\ell)\,b + (3\ell-k+1)\,a,$$

where

$$\left(\mathcal{E}^{(p)} + x\right)^n = E_n^{(p)}(x), \quad \left(\mathcal{B}^{(p)} + x\right)^n = B_n^{(p)}(x), \quad \mathcal{U}^n = \frac{1}{n+1}, \quad \mathcal{U}^{(p)} = \mathcal{U}_1 + \cdots + \mathcal{U}_p.$$

- Other examples, models, etc.

Thank you for your attention