Random Walk Models for Identities Involving Bernoulli and Euler Polynomials of Higher-orders

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Motivation

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Definition

The **Euler numbers** E_n, **Euler polynomials** $E_n(x)$, and **Euler polynomials of order** p**,** $E_n^{(p)}(x)$, are defined via their (exponential) generating functions

$$\frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad \frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad \left(\frac{2}{e^t+1}\right)^p e^{xt} = \sum_{n=0}^{\infty} E_n^{(p)}(x) \frac{t^n}{n!}.$$

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Fact: Convolution

$$E_n^{(p)}(x) = \sum_{k_1+\cdots+k_p=n} \binom{n}{k_1,\ldots,k_p} E_{k_1}(x) E_{k_2}(0) \cdots E_{k_p}(0).$$

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For any positive integer N,

$$E_n(x) = \frac{1}{N^n} \sum_{\ell=N}^{\infty} p_\ell^{(N)} E_n^{(\ell)}\left(\frac{\ell - N}{2} + Nx\right),$$

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Namely, T_n is the Chebyshev polynomial of the 1st kind.

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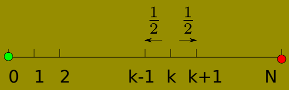

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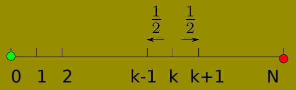

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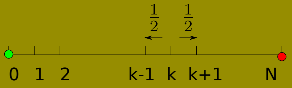

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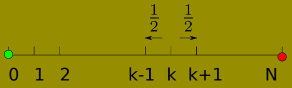

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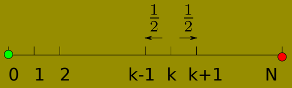

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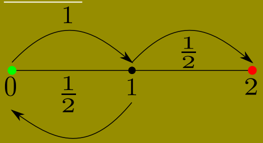

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$$E_n(x) = \mathbb{E}\left[\left(x + iL - \frac{1}{2}\right)^n\right] = \int_{\mathbb{R}} \left(x + it - \frac{1}{2}\right)^n \operatorname{sech}(\pi t)\, \mathrm{d}t.$$

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Theorem (Klebanov et al.)

The random variable

$$Z_N = \frac{1}{N}\sum_{j=1}^{\nu_N} L_j$$

has the same hyperbolic secant distribution (as L_j's).

L. B. Klebanov, A. V. Kakosyan, S. T. Rachev, and G. Temnov. On a class of distributions stable under random summation. *J. Appl. Prob.*, **49** (2012), 303–318.

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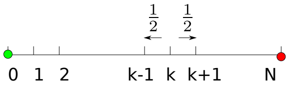
  $$\mathbb{E}\left[\left(i\sum_{j=1}^{\ell} L_j - \frac{\ell}{2} + Nx - \frac{N}{2} + \frac{\ell}{2}\right)^n\right] = E_n^{(\ell)}\left(\frac{\ell - N}{2} + Nx\right).$$

□

Reflected Brownian Motion



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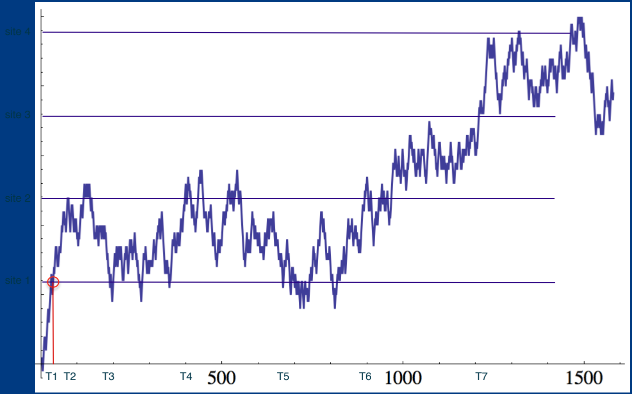

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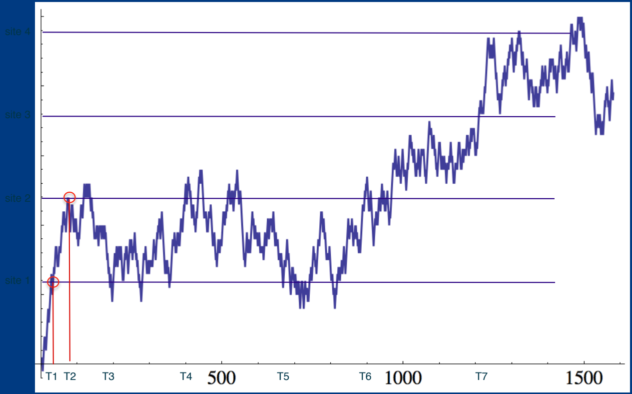

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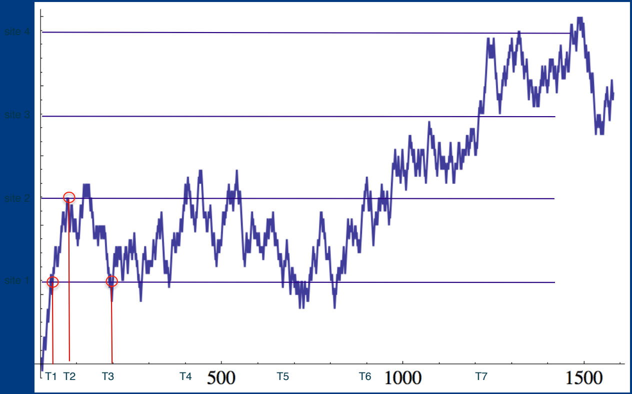

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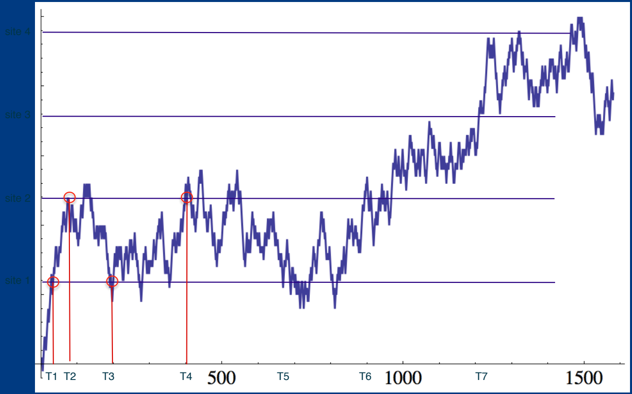

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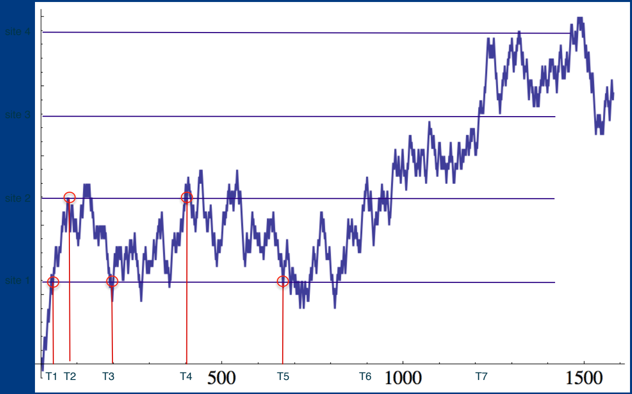

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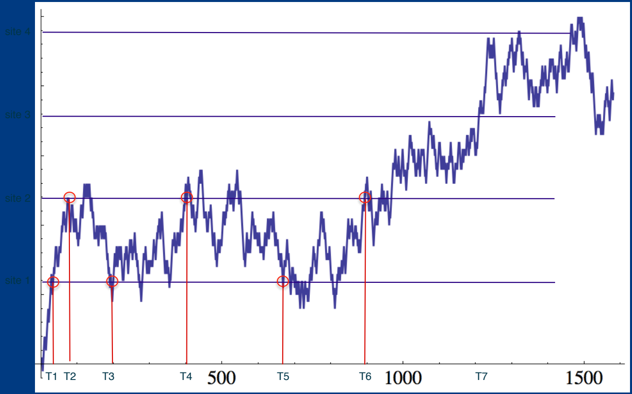

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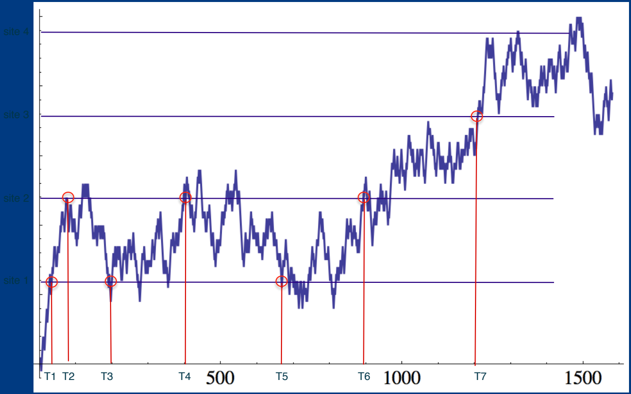

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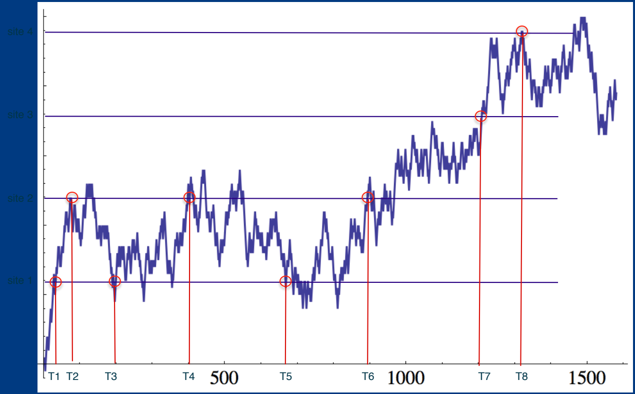

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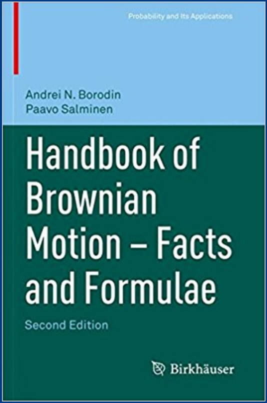

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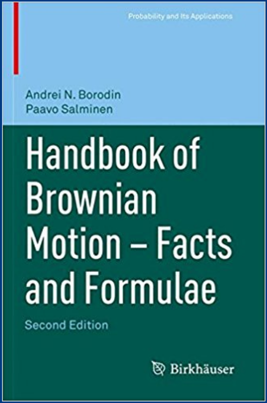

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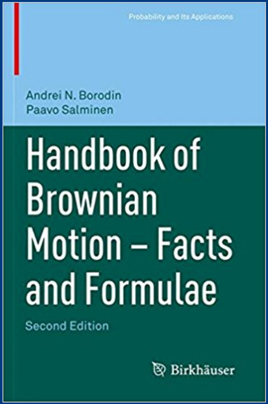

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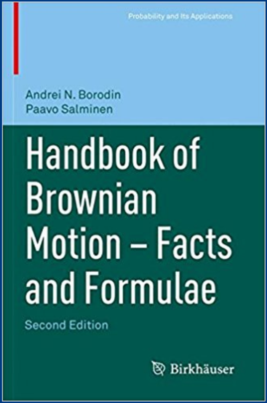

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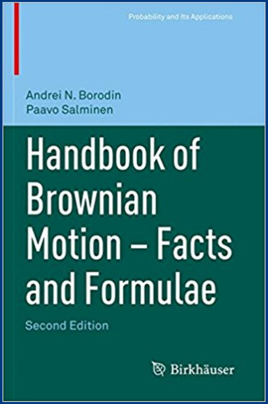

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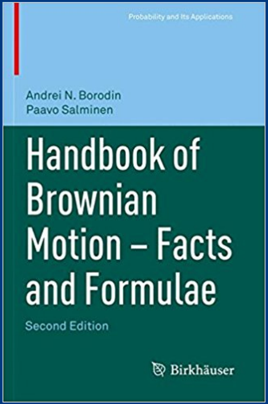

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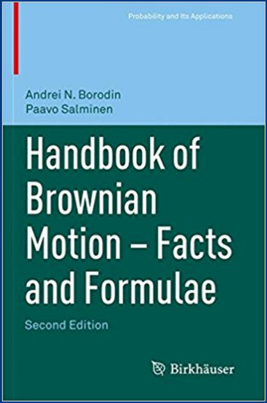

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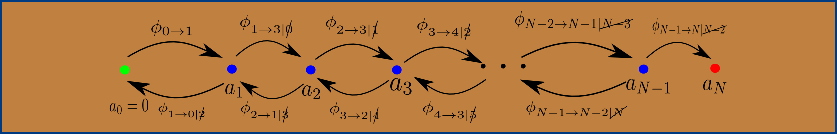
3.

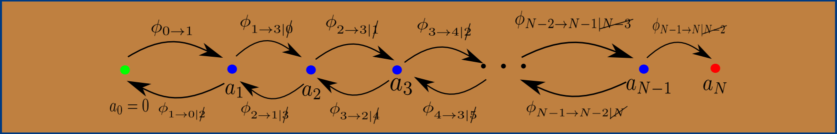
$$\mathbb{E}\left[e^{s\left(iL-\frac{1}{2}\right)}\right] = \int_{\mathbb{R}} \frac{e^{s\left(it-\frac{1}{2}\right)}}{\cosh(\pi t)}\mathrm{d}t = \frac{e^{-\frac{s}{2}+sx}}{\cosh\left(\frac{s}{2}\right)}e^{sx}.$$

$$\frac{2}{1+e^s}e^{sx} = \sum_{n=0}^{\infty} E_n(x)\frac{s^n}{n!}$$

Idea

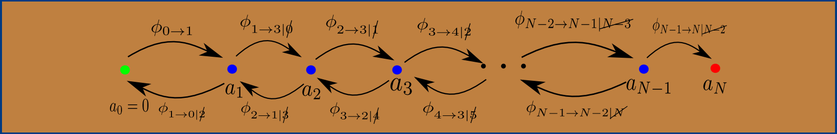

$\phi_{0\to1}$
$\phi_{1\to3|\not0}$
$\phi_{2\to3|\not1}$
$\phi_{3\to4|\not2}$
$\phi_{N-2\to N-1|\not{N-3}}$
$\phi_{N-1\to N|\not{N-2}}$
$a_0 = 0$
a_1
a_2
a_3
a_{N-1}
a_N
$\phi_{1\to0|\not2}$
$\phi_{2\to1|\not3}$
$\phi_{3\to2|\not4}$
$\phi_{4\to3|\not5}$
$\phi_{N-1\to N-2|\not N}$

Idea



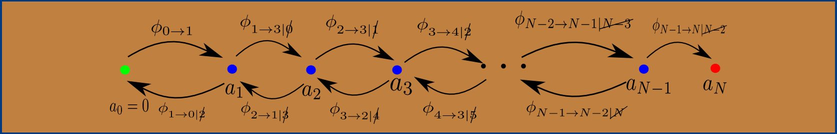
- A reflected Brownian motion model with N equally distributed sites; $a_j = j$ or j/N;

Idea



- A reflected Brownian motion model with N equally distributed sites; $a_j = j$ or j/N;
- (modified) hitting times with "step" from one site to a nearby site .

Idea



- A reflected Brownian motion model with N equally distributed sites; $a_j = j$ or j/N;
- (modified) hitting times with "step" from one site to a nearby site .

$$\mathbb{E}_x\left[e^{-\alpha H_z}\right] = \begin{cases} \frac{\cosh(xw)}{\cosh(zw)}, & 0 \le x \le z; \\ e^{-(x-z)w}, & z \le x. \end{cases}$$

1-dim, 1-loop

With $p \leq q \leq r$, $w = \sqrt{2\alpha}$

$$\phi_{p\to q} := \mathbb{E}_p\left[e^{-\alpha H_q}\right] = \frac{\cosh(pw)}{\cosh(qw)},$$

$$\phi_{q\to p|\not{r}} := \mathbb{E}_q\left[e^{-\alpha H_p}|W_t < r\right] = \frac{\sinh((r-q)\,w)}{\sinh((r-p)\,w)},$$

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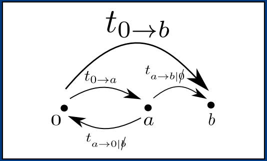

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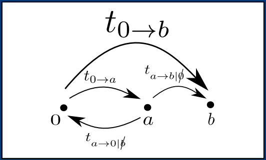

1-dim, 1-loop

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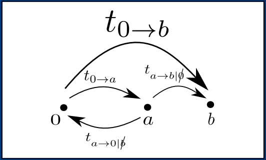

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1-dim, 1-loop

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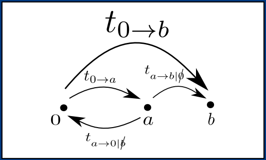

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- Generating functions:

$$\phi_{0\to b} = \phi_{0\to a}\phi_{a\to b|\not 0}\sum_{\ell=0}^{\infty}\left(\phi_{0\to a}\phi_{a\to 0|\not b}\right)^{\ell}$$

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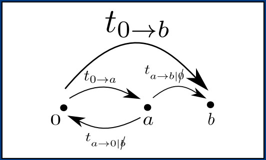

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$$\operatorname{sech}(bw) \quad = \quad \operatorname{sech}(aw)\cdot\frac{\sinh(aw)}{\sinh(bw)}\sum_{\ell=0}^{\infty}\left[\operatorname{sech}(aw)\cdot\frac{\sinh((b-a)w)}{\sinh(bw)}\right]^{\ell}$$

1-dim, 1-loop

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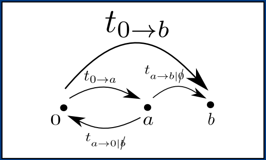

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- Generating functions:

$$\phi_{0\to b} = \phi_{0\to a}\phi_{a\to b|\not 0}\sum_{\ell=0}^{\infty}\left(\phi_{0\to a}\phi_{a\to 0|\not b}\right)^{\ell}$$

$$\begin{aligned}\operatorname{sech}(bw) &= \operatorname{sech}(aw)\cdot\frac{\sinh(aw)}{\sinh(bw)}\sum_{\ell=0}^{\infty}\left[\operatorname{sech}(aw)\cdot\frac{\sinh((b-a)w)}{\sinh(bw)}\right]^{\ell} \\ &= \operatorname{sech}(aw)\cdot\frac{\sinh(aw)}{\sinh(bw)}\cdot\frac{1}{1-\operatorname{sech}(aw)\cdot\frac{\sinh((b-a)w)}{\sinh(bw)}}\end{aligned}$$

1-dim, 1-loop

Prop. (**LJ** and C. Vignat, 2018)

$$E_n\left(\frac{x}{2b}+\frac{3}{2}-2\frac{a}{b}\right)-E_n\left(\frac{x}{b}+\frac{1}{2}\right)=\frac{(n+1)\left(1-2\frac{a}{b}\right)2^n a^n}{b^n}\sum_{\ell=0}^{\infty}\frac{\frac{a}{b}\left(1-\frac{a}{b}\right)^{\ell}}{}B_n^{(\ell+1)}\left(\frac{x+b}{4a}+\frac{\ell}{2}\right).$$

1-dim, 1-loop

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- $\frac{a}{b}\left(1-\frac{a}{b}\right)^{\ell}$ are the probability weights of a geometric distribution with parameter a/b.

1-dim, 1-loop

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- $\frac{a}{b}\left(1-\frac{a}{b}\right)^{\ell}$ are the probability weights of a geometric distribution with parameter a/b.

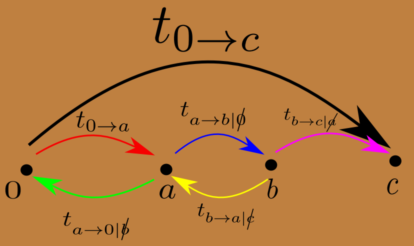
Definition

The **Bernoulli numbers** B_n, **Bernoulli polynomials** $B_n(x)$, and **Bernoulli polynomials of order** p, $B_n^{(p)}(x)$, are defined via their (exponential) generating functions

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!},\quad \frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\quad \left(\frac{t}{e^t-1}\right)^p e^{xt}=\sum_{n=0}^{\infty}B_n^{(p)}(x)\frac{t^n}{n!}.$$

1-dim, 2-loops

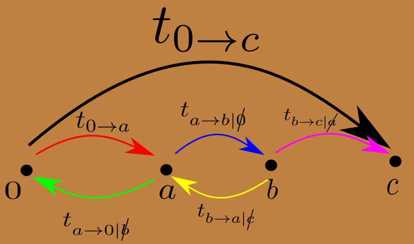
How about 2-loops?



t, t, t, t, t, t

1-dim, 2-loops

How about 2-loops?



t, t, t, t, t, t

$t \;=$

1-dim, 2-loops

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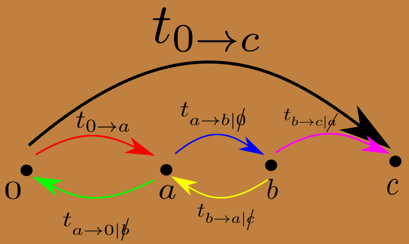

t, t, t, t, t, t

$t = t$

1-dim, 2-loops

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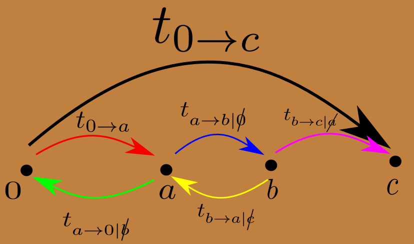
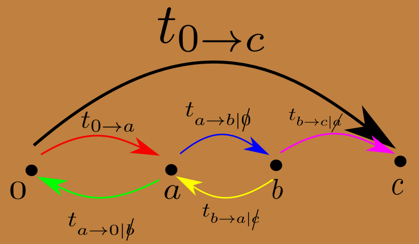


$t, \; t, \; t, \; t, \; t, \; t$

$t \;=\; t + t$

1-dim, 2-loops

How about 2-loops?

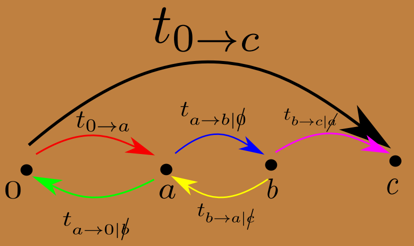


t, t, t, t, t, t

$$t = t + t + t$$

1-dim, 2-loops

How about 2-loops?



t, t, t, t, t, t

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1-dim, 2-loops

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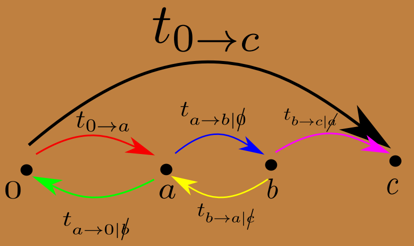

$t,\ t,\ t,\ t,\ t,\ t$

$$t \quad = \quad t + t + t + t + t$$

1-dim, 2-loops

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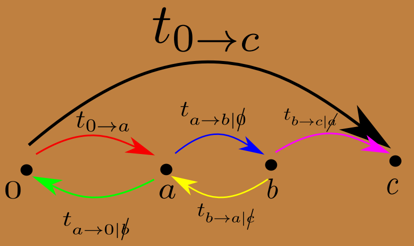

$t, \ t, \ t, \ t, \ t, \ t$

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1-dim, 2-loops

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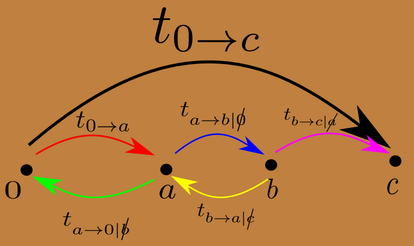

$t, \; t, \; t, \; t, \; t, \; t$

$$t \quad = \quad t + t + t + t + t + t + \cdots$$

1-dim, 2-loops

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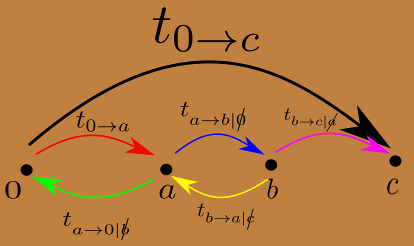

$t,\ t,\ t,\ t,\ t,\ t$

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1-dim, 2-loops

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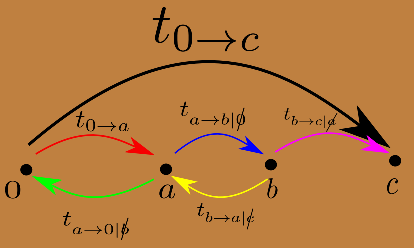
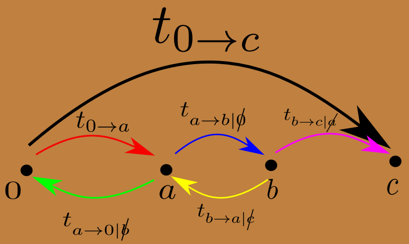


$t, \; t, \; t, \; t, \; t, \; t$

$$
\begin{aligned}
t &= t + t + t + t + t + t + \cdots + t + t \\
&= t + t + t + \underbrace{(t + t) + \cdots + (t + t)}_{k \text{ loops}} + \underbrace{(t + t) + \cdots + (t + t)}_{\ell \text{ loops}}
\end{aligned}
$$

1-dim, 2-loops

How about 2-loops?



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We can generalize it to n-loop model.

1-dim, 2-loops

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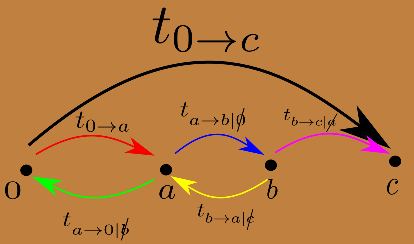

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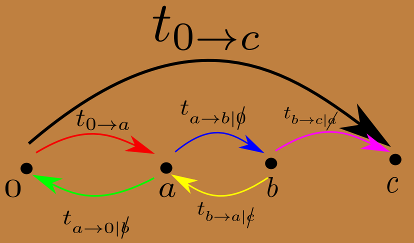
\end{aligned}
$$

We can generalize it to n-loop model.

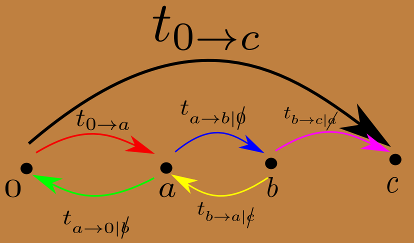
Unfortunately, this is WRONG.....

1-dim, 2-loops



$$\phi = \phi \cdot \phi \cdot \phi \cdot \left[\sum_{k=0}^{\infty} (\phi\phi)^k\right]\left[\sum_{\ell=0}^{\infty} (\phi\phi)^\ell\right] = \frac{\phi \cdot \phi \cdot \phi}{(1-\phi\phi)(1-\phi\phi)}$$

1-dim, 2-loops



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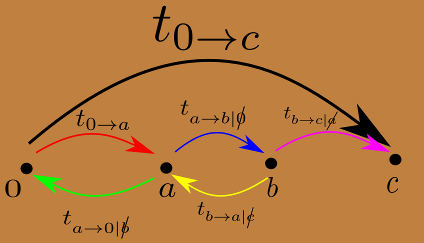
Let $a = 1$, $b = 2$ and $c = 3$.

1-dim, 2-loops



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Let $a = 1$, $b = 2$ and $c = 3$.

$$\begin{aligned}
\text{LHS} &= \phi = \phi_{\mathbf{0}\to\mathbf{3}} = \operatorname{sech}(3w) = \frac{1}{\cosh(3w)} \\
\text{RHS} &= \frac{\frac{1}{\cosh(w)}\cdot\frac{\sinh(w)}{\sinh(2w)}\cdot\frac{\sinh(w)}{\sinh(2w)}}{\left(1-\frac{1}{\cosh(w)}\frac{\sinh(w)}{\sinh(2w)}\right)\left(1-\frac{\sinh(w)}{\sinh(2w)}\frac{\sinh(w)}{\sinh(2w)}\right)} = \frac{\frac{1}{4\cosh^3 w}}{\left(1-\frac{1}{2\cosh^2 w}\right)\left(1-\frac{1}{4\cosh^2 w}\right)} \\
&= \frac{2\cosh w}{(2\cosh^2 w-1)\,(4\cosh^2 w-1)} \neq \frac{1}{\cosh(3w)} = \frac{1}{4\cosh^3 w-3\cosh w}
\end{aligned}$$

Two-loops

$$I := \phi_{a\to b|\not{A}}\,\phi_{b\to a|\not{c}}, \qquad II := \phi_{b\to c|\not{a}}\,\phi_{c\to b|\not{B}}$$

- k loops of I followed by l loops of II, with $k, l = 0, 1, \ldots,$ which gives

$$\sum_{k,l} I^k II^l = \frac{1}{1-I}\cdot\frac{1}{1-II};$$

- k_1 loops of I followed by l_1 loops of II, then followed by k_2 loops of I and finally followed by l_2 loops of II, with k_1, l_2 nonnegative and k_2, l_1 positive, which gives

$$\sum_{k_1,l_2=0,k_2,l_1=1}^{\infty} I^{k_1} II^{l_1} I^{k_2} II^{l_2} = \frac{I\cdot II}{(1-I)^2\,(1-II)^2};$$

- the general term will be k_1 loops of $I \to l_1$ loops of $II \to \cdots \to k_n$ loops of $I \to l_n$ loops of II, with k_1, l_n nonnegative and the rest indices positive, which gives

$$\frac{(I\cdot II)^{n-1}}{(1-I)^n\,(1-II)^n}.$$

Therefore, loops I and II contribute as

$$\sum_{n=1}^{\infty}\frac{(I\cdot II)^{n-1}}{(1-I)^n\,(1-II)^n} = \frac{1}{1-(I+II)} = \sum_{k=0}^{\infty}(I+II)^k.$$

Two Loops

Prop. (LJ. and C. Vignat, 2018)

For any positive integer n,

$$E_n\left(\frac{x}{6}\right) = \sum_{k=0}^{\infty} \frac{3^{k-n}}{4^{k+1}} E_n^{(2k+3)}\left(\frac{x}{2}+k\right).$$

Two Loops

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$$E_n\left(\frac{x}{6}\right) = \sum_{k=0}^{\infty} \frac{3^{k-n}}{4^{k+1}} E_n^{(2k+3)}\left(\frac{x}{2}+k\right).$$

In general

$$(x+2c\mathcal{E}+c)^n = \sum_{k=0}^{\infty}\sum_{\ell=0}^{k} q_{k,\ell}\Big[x+2(b-a)\mathcal{B}+2(c-b)\mathcal{B}'+a\mathcal{E}^{(\ell)}+2(b-a)\mathcal{U}^{(l)} + 2a\mathcal{U}'^{(k-\ell)}+2(b-a)\mathcal{B}'^{(k-\ell)}+q'_{k,\ell}\Big]^n,$$

$$q_{k,\ell} := \binom{k}{\ell}\frac{(b-a)^{\ell+1}a^{k-\ell+1}(c-b)^{k-\ell}}{b^{k+1}(c-a)^{k-\ell+1}} \qquad q'_{k,\ell} = c+(2k-2\ell)b+(3\ell-k+1)a,$$

where

$$\left(\mathcal{E}^{(p)}+x\right)^n = E_n^{(p)}(x), \quad \left(\mathcal{B}^{(p)}+x\right)^n = B_n^{(p)}(x), \quad \mathcal{U}^n = \frac{1}{n+1}, \quad \mathcal{U}^{(p)} = \mathcal{U}_1+\cdots+\mathcal{U}_p.$$

n loops?

Consider consecutive loops l_1, l_2, ..., l_n, it seems like the contribution is

$$\sum_{k=0}^{\infty}\left(\sum_{\ell=1}^{n} l_\ell\right)^k = \frac{1}{1-(l_1+\cdots+l_n)}. \qquad (*)$$

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- In general sites $0, 1, \ldots, N$:

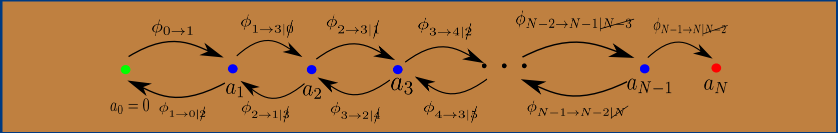
$$\frac{1}{\cosh(Nw)} \overset{??}{=} \frac{\frac{1}{\cosh w}\cdot\left(\frac{\sinh w}{\sinh(2w)}\right)^N}{1-\left(\frac{1}{\cosh(w)}\frac{\sinh(w)}{\sinh(2w)}+(N-1)\frac{\sinh(w)}{\sinh(2w)}\frac{\sinh(w)}{\sinh(2w)}\right)}$$

$$= \frac{\frac{1}{2^N\cosh^{N+1} w}}{1-\frac{N+3}{4}\cosh^N w}.$$

This shows $(*)$ is not correct.

$$\frac{1}{\cosh(Nw)} = \frac{1}{\cos(Niw)} = \frac{1}{T_N(\cos(iw))} = \frac{1}{T_N(\cosh w))}.$$

General Formula



Thm. (**LJ**, I. Simonelli, and H. Yue, 2022)

$$\phi_{0\to a_n} = \phi_{0\to a_1}\phi_{a_1\to a_2|\not{0}}\cdots\phi_{a_{n-1}\to a_n|\not{a_{n-2}}}\cdot\frac{1}{1-P(L_1,\ldots,L_n)},$$

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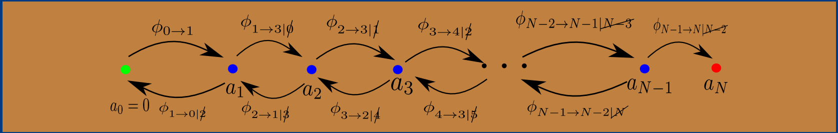

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$$\phi_{0\to a_n} = \phi_{0\to a_1}\phi_{a_1\to a_2|\cancel{0}}\cdots\phi_{a_{n-1}\to a_n|\cancel{a_{n-2}}}\cdot\frac{1}{1-P(L_1,\ldots,L_n)},$$

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$$L_j = \phi_{a_{j-1}\to a_j|\cancel{a_{j-2}}}\phi_{a_j\to a_{j-1}|\cancel{a_{j+1}}}$$

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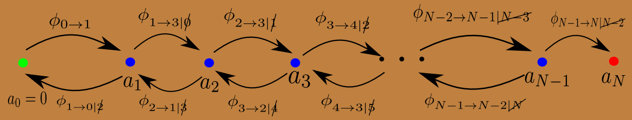

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for the condition $*$ given by

- $\ell = 1,2,\dots,n$;
- $j_1 < j_2-1$, $j_2<j_3-1$,…,$j_{\ell-1}<j_\ell-1$.

Examples

- $n = 2$: $P = L_1 + L_2$;

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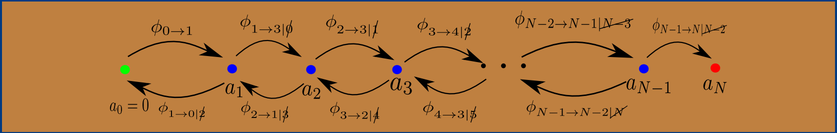

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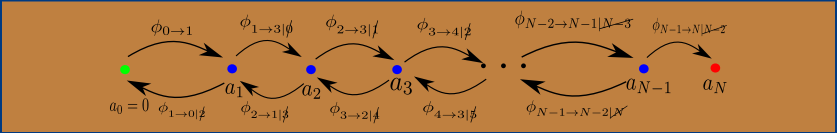

1. Induction.

Examples

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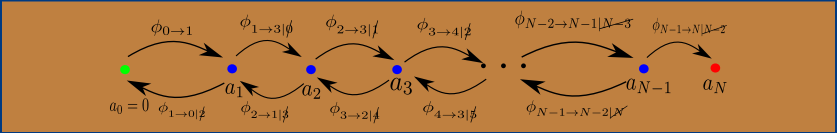

1. Induction. The tricky part is, if we "glue" the first two loops together; or ignore site a_1, $\phi_{2\to 3|\not 1}$ should be replaced by $\phi_{2\to 3|\emptyset}$.

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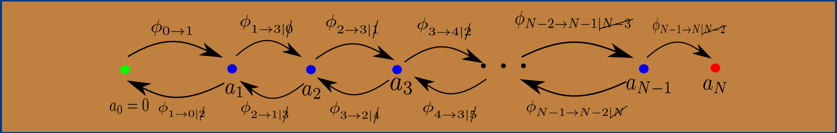

1. Induction. The tricky part is, if we "glue" the first two loops together; or ignore site a_1, $\phi_{2\to 3|\not 1}$ should be replaced by $\phi_{2\to 3|\not 0}$.
2. Inclusion-exclusion principle

Inclusion-exclusion principle

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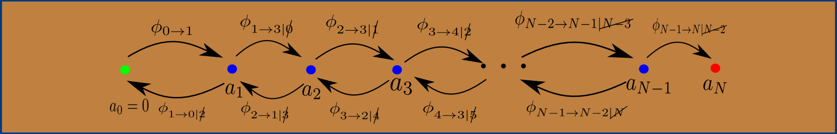

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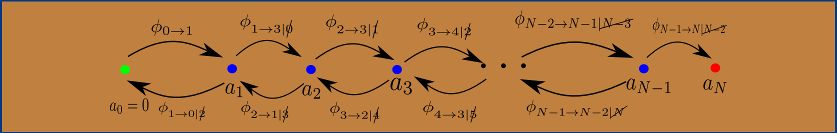


$$\frac{1}{1-(L_1+L_2+L_3-L_3\cdot L_1)} = \sum_{k=0}^{\infty} (L_1+L_2+L_3-L_3\cdot L_1)^k \,.$$

Results

Thm. (LJ, I. Simonelli, and H. Yue, 2022)

By letting $a_j = j$, we have

$$E_n(x) = \frac{1}{4^n} \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \frac{2^n}{8^{\ell+1}} E_n^{(2k+2\ell)}\left(4x + k + \ell\right).$$

$$E_n\left(x\right) = \sum_{k=0}^{\infty} \frac{5^{k-n}}{4^{k+\ell+2}} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} E_n^{(2\ell+2k+5)}\left(5x + \ell + k\right).$$

$$\cdots$$

Generalization

- Bessel process in $\mathbb{R}^n$:

$$R_t^{(n)} := \sqrt{\left(W_t^{(1)}\right)^2 + \cdots + \left(W_t^{(n)}\right)^2}$$

- Moment generating functions for hitting times:
  $H_z := \min_s \left\{ R_s^{(n)} = z \right\}$.

$$\mathbb{E}_x \left(e^{-\alpha H_z}; \sup_{0 \le s \le H_z} R_s^{(n)} < y \right) = \begin{cases} \frac{x^{-\nu} I_\nu(xw)}{z^{-\nu} I_\nu(zw)}, & 0 \le x \le z \le y; \\ \frac{S_\nu(yw, xw)}{S_\nu(yw, zw)}, & z \le x \le y, \end{cases}$$

- $n = 2 + 2\nu$ for $\nu \ge 0$

$$S_\nu(x, y) := (xy)^{-\nu} \left[I_\nu(x) K_\nu(y) - K_\nu(x) I_\nu(y) \right],$$

and

$$I_\nu(x) = \sum_{\ell=0}^{\infty} \frac{1}{\ell! \Gamma(\ell + \nu + 1)} \left(\frac{x}{2}\right)^{2\ell + \nu}, \quad K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\nu\pi)}.$$

$n = 3 \Leftrightarrow \nu = 1/2$

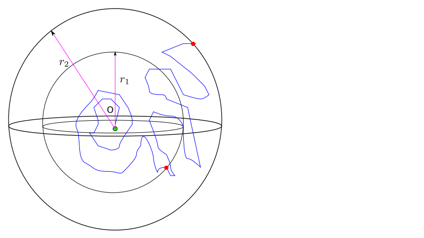
▶

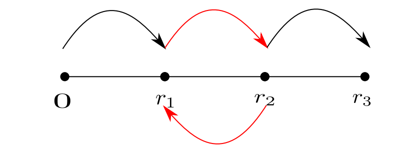
$$I_{\frac{1}{2}}(x) = \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2m+\frac{1}{2}}}{m!\Gamma\left(m+\frac{3}{2}\right)} = \sqrt{\frac{2}{x\pi}}\sinh(x)$$

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t - 1}e^{tx} = \frac{te^{tx}e^{-\frac{t}{2}}}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} = \frac{te^{t\left(x-\frac{1}{2}\right)}}{2}\sinh\left(\frac{t}{2}\right)$$

$$K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}$$

$$\mathbb{E}_x\left(e^{-\alpha H_z};\ \sup_{0\le s\le H_z} R_s^{(3)} < y\right) = \begin{cases} \frac{z\sinh(xw)}{x\sinh(zw)}, & 0 \le x \le z \le y \\ \frac{z\sinh((y-x)w)}{x\sinh((y-z)w)}, & z \le x \le y \end{cases}$$





$n = 3 \Leftrightarrow \nu = 1/2, r_1 = 1, r_2 = 2, r_3 = 3$

Prop. (LJ. and C. Vignat, 2018)

$$\frac{3^{n+1}}{n+1}\left[B_{n+1}\left(\frac{x}{6}+\frac{5}{6}\right) - B_{n+1}\left(\frac{x}{6}+\frac{1}{2}\right)\right] = \sum_{k\geq 0}\frac{3}{4}\left(\frac{1}{4}\right)^k E_n^{(2k+2)}\left(\frac{x+3+2k}{2}\right).$$

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Corollary

1. *Take* $x = 0$, $n = 2m-1$ *in* ($\circledast$).

$$B_{2m} = \frac{m}{(1-2^{1-2m})(3^{2m}-1)}\sum_{k\geq 0}\left(\frac{1}{4}\right)^k E_{2m-1}^{(2k+2)}\left(k+\frac{3}{2}\right).$$

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2. *Take* $n = 1$ *in* $(\circledast)$.

$$\sum_{k\geq 0}\frac{3}{4}\left(\frac{1}{4}\right)^k\left(\frac{x+3+2k}{2} - k - 1\right) = \sum_{k\geq 0}\frac{3}{4}\left(\frac{1}{4}\right)^k\left(\frac{x+1}{2}\right) = \frac{x+1}{2}.$$

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Prop. (**LJ** and C. Vignat, 2018)

For any positive integer n,

$$3^n B_n\left(\frac{x+4}{6}\right) = \sum_{k=0}^{\infty}\frac{1}{2^k}E_n^{(2k+2)}\left(\frac{x+2k+3}{2}\right).$$

Results

Thm. (LJ, I. Simonelli, and H. Yue, 2022)

For any $m \in \mathbb{N}$and let $M = m - 1$ and M' be the largest odd number less than or equal to M, then

$$
\begin{aligned}
& B_{n+1}\left(\frac{2+x}{m+2}\right) - B_{n+1}\left(\frac{x}{m+2}\right) \\
&= \frac{n+1}{(m+2)^n} \sum_{k=0}^{\infty} \frac{m^k}{2^{2k+m}} \sum_{n_1,\ldots,n_M=0}^{k} \binom{k}{n_1,\ldots,n_M} (-1)^{n_1+n_3+\cdots+n_{M'}} \\
& \times \frac{\binom{m-1}{2}^{n_1}\binom{m-2}{3}^{n_2}\cdots\binom{m-M}{M}^{n_M}}{4^{n_1+2n_2+Mn_M}\, m^{n_1+\cdots+n_M}} \\
& \times E_n^{(2k+2n_1+4n_2+\cdots+2Mn_M+m)}(k+n_1+2n_2+\cdots+Mn_M+x).
\end{aligned}
$$

Example

$$B_{n+1}\left(\frac{x+2}{5}\right) - B_{n+1}\left(\frac{x}{5}\right) = \frac{n+1}{5^n}\sum_{k=0}^{\infty}\frac{3^k}{2^{2k+3}}\sum_{\ell=0}^{k}\binom{k}{\ell}\frac{(-1)^\ell}{12^\ell}E_n^{(2k+2\ell+3)}(k+\ell+x).$$

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$$\sum_{n=0}^{\infty}\left(B_{n+1}\left(\frac{x+2}{5}\right) - B_{n+1}\left(\frac{x}{5}\right)\right)\frac{t^{n+1}}{(n+1)!}$$

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$$= \quad \frac{t\left(e^{\frac{2t}{5}}-1\right)}{e^t-1}e^{\frac{tx}{5}}$$

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&= \frac{t\left(e^{\frac{2t}{5}}-1\right)}{e^t-1}e^{\frac{tx}{5}} \\
&= t\sum_{n=0}^{\infty}\left(\frac{1}{5^n}\sum_{k=0}^{\infty}\frac{3^k}{2^{2k+3}}\sum_{\ell=0}^{k}\binom{k}{\ell}\frac{(-1)^\ell}{12^\ell}E_n^{(2k+2\ell+3)}(k+\ell+x)\right)\frac{t^n}{n!}.
\end{aligned}$$

General formula

2-loop, 3-dim

$$(x + 2c\mathcal{E} + c)^n = \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} q_{k,\ell} \left[x + 2\,(b-a)\,\mathcal{B} + 2\,(c-b)\,\mathcal{B}' + a\mathcal{E}^{(\ell)} + 2\,(b-a)\,\mathcal{U}^{(l)} \right.$$
$$\left. + 2a\mathcal{U}'^{(k-\ell)} + 2\,(b-a)\,\mathcal{B}'^{(k-\ell)} + q'_{k,\ell} \right]^n,$$

$$q_{k,\ell} := \binom{k}{\ell} \frac{(b-a)^{\ell+1}\, a^{k-\ell+1}\, (c-b)^{k-\ell}}{b^{k+1}\,(c-a)^{k-\ell+1}} \qquad q'_{k,\ell} = c + (2k - 2\ell)\, b + (3\ell - k + 1)\, a,$$

where

$$\left(\mathcal{E}^{(p)} + x\right)^n = E_n^{(p)}(x), \quad \left(\mathcal{B}^{(p)} + x\right)^n = B_n^{(p)}(x), \quad \mathcal{U}^n = \frac{1}{n+1}, \quad \mathcal{U}^{(p)} = \mathcal{U}_1 + \cdots + \mathcal{U}_p.$$

Uniqueness

Prop. (LJ. and C. Vignat, 2018)

$$B_n\left(\frac{x+4}{6}\right) = \frac{1}{3^n}\sum_{k=0}^{\infty}\frac{E_n^{(2k+2)}\left(\frac{x+2k+3}{2}\right)}{2^k}$$

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Prop. (LJ. and C. Vignat, 2018)

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$$\begin{aligned}\frac{4w}{\sinh(4w)} &= \frac{w}{\sinh w}\,\frac{2\sinh w}{\sinh(2w)}\,\frac{3\sinh w}{2\sinh(2w)}\,\frac{4\sinh w}{3\sinh(2w)}\sum_{k=0}^{\infty}\frac{\operatorname{sech}^{2k} w}{2^k}\\ &= \frac{w}{\sinh w}\sum_{k=0}^{\infty}\frac{\operatorname{sech}^{2k+3} w}{2^{k+1}}\end{aligned}$$

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Thank you for your attention