Hankel Determinants on Bernoulli polynomials and q-analogues

Lin Jiu

Sept. 1st, 2023

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Given a sequence (a_k), the nth Hankel determinant is defined by

$$\det \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{pmatrix}.$$

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$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{t}{e^t - 1} e^{xt}.$$

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And $B_n = B_n(0)$ is the Bernoulli number.

- **Hankel determinants of certain sequences of Bernoulli polynomials: A direct proof of an inverse matrix entry from Statistics**
  Lin Jiu and Ye Li
  To Appear in Contributions to Discrete Mathematics

- **Hankel Determinants of shifted sequences of Bernoulli and Euler numbers**
  Karl Dilcher and Lin Jiu
  To Appear in Contributions to Discrete Mathematics

- **Compatibility of the method of brackets with classical integration rules** [url]
  Zachary Bradshaw, Ivan Gonzalez, Lin Jiu, Victor Hugo Moll, and Christophe Vignat
  Open Mathematics 21(1), Article number: 20220581, 2023.

- **Moments and cumulants on identities for Bernoulli and Euler numbers** [url]
  Lin Jiu and Diane Yahui Shi
  Mathematical Reports 24(4), pp. 643–650, 2022

- **Loop Decompositions of Random Walks and Nontrivial Identities of Bernoulli and Euler Polynomials** [url]
  Lin Jiu, Italo Simonelli, and Heng Yue
  INTEGERS 22, Article 91, 2022

- **Hankel Determinants of sequences related to Bernoulli and Euler Polynomials** [url]
  Karl Dilcher and Lin Jiu
  International Journal of Number Theory, 18(2) pp. 331--359, 2022.

- **Orthogonal Polynomials and Hankel Determinants for Certain Bernoulli and Euler Polynomials** [url]
  Karl Dilcher and Lin Jiu
  Journal of Mathematical Analysis and Applications, 497(1), Article 124855, 2021

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- $$B_n'(x) = nB_{n-1}(x) \Leftrightarrow \frac{d}{dx}(\mathcal{B}+x)^n = n(\mathcal{B}+x)^{n-1}.$$

Probabilistic Interpretation

Theorem

$$B_n = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} z^n \left(\frac{\pi}{\sin(\pi z)} \right)^2 dz.$$

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Let L_B be a random variable with density $\pi \operatorname{sech}^2(\pi x)/2$, then $\mathcal{B} = iL_B - 1/2$. In particular,

$$(\mathcal{B} + x)^n = \mathbb{E}\left[(\mathcal{B} + x)^n\right] = B_n(x).$$

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Theorem

A sequence of numbers a_n is the sequence of moments of a measure μ if and only if a certain positivity condition is fulfilled; namely, the Hankel matrices

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \end{pmatrix}$$

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The (monic) orthogonal polynomails w.r.t. a sequence a_n can be defined by

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$$P_{n+1}(y) = (y+\alpha_n)P_n(y) - \beta_n P_{n-1}(y).$$

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$$H_n(B_k) = (-1)^{\binom{n+1}{2}} \prod_{j=1}^{n} \frac{j!^6}{(2j)!(2j+1)!}$$

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Polynomial?

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Theorem

If $c_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k c_{n-k}$*, then* $H_n(c_k) = H_n(c_k(x))$.

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Table 2
$b_n^{(p)}$ for $1 \leq n,\ p \leq 5$.

	$p = 1$	$p = 2$	$p = 3$	$p = 4$	$p = 5$
$n = 1$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{5}{12}$
$n = 2$	$\frac{4}{15}$	$\frac{13}{30}$	$\frac{3}{5}$	$\frac{23}{30}$	$\frac{14}{15}$
$n = 3$	$\frac{81}{140}$	$\frac{372}{455}$	$\frac{1339}{1260}$	$\frac{2109}{1610}$	$\frac{1527}{980}$
$n = 4$	$\frac{64}{63}$	$\frac{3736}{2821}$	$\frac{138688}{84357}$	$\frac{668543}{339549}$	$\frac{171830}{74823}$
$n = 5$	$\frac{625}{396}$	$\frac{1245075}{636988}$	$\frac{299594775}{127670972}$	$\frac{42601023200}{15509529057}$	$\frac{3638564965}{1154491404}$

Euler Case

Definition

The generalized Euler polynomial $E_n(x)$ of order p is defined by its exponential generating function

$$\sum_{n=0}^{\infty} E_n^{(p)}(x)\frac{t^n}{n!} = \left(\frac{2}{e^t+1}\right)^p e^{xt}.$$

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$$\sum_{n=0}^{\infty} E_n^{(p)}(x)\frac{t^n}{n!} = \left(\frac{2}{e^t+1}\right)^p e^{xt}.$$

Theorem (L. J and D.Y.H. Shi)

let $\Omega_n^{(p)}(y)$ be the monic orthogonal polynomials with respect to $E_n^{(p)}(x)$. Then

$$\Omega_{n+1}^{(p)}(y) = \left(y - x + \frac{p}{2}\right)\Omega_n^{(p)}(y) + \frac{n(n+p-1)}{4}\Omega_{n-1}^{(p)}(y).$$

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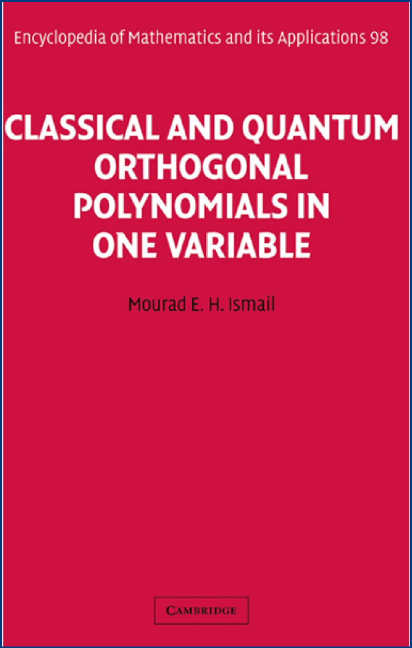
$$\Omega_n^{(p)}(y) = \frac{i^n n!}{2^n} P_n^{(\frac{p}{2})}\left(-i\left(y - x + \frac{p}{2}\right); \frac{\pi}{2}\right) \text{—Maxiner-Pollaczek.}$$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CLASSICAL AND QUANTUM ORTHOGONAL POLYNOMIALS IN ONE VARIABLE
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A summation on Bernoulli numbers

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Corollary 5.6.

$$\det_{0\leqslant i,j\leqslant n}\left(B_{2i+2j}\left(\tfrac{1}{2}\right)\right)=\prod_{i=1}^{n}\left(\frac{(2i-1)^4 i^4}{(4i-3)(4i-1)^2(4i+1)}\right)^{n-i+1}. \qquad (41)$$

With Karl

Theorem (K. Dilcher and L. J)

$$H_n\left(B_{2k+1}\left(\frac{x+1}{2}\right)\right) = (-1)^{\binom{n+1}{2}}\left(\frac{x}{2}\right)^{n+1}\prod_{\ell=1}^{n}\left(\frac{\ell^4(x^2-\ell^2)}{4(2\ell+1)(2\ell-1)}\right)^{n+1-\ell}.$$

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Fact

Sequence a_k, Orthogonal Polynomials $P_n(y)$ with

$$P_{n+1}(y) = (y+\alpha_n)P_n(y) - \beta_n P_{n-1}(y)$$

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1. $H_n(a_k) = a_0^{n+1}\beta_1^n\cdots\beta_n.$

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1. $H_n(a_k) = a_0^{n+1}\beta_1^n \cdots \beta_n.$
2. $$\sum_{k=0}^{\infty} a_k z^k =$$

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Theorem (K. Dilcher and L. J)

$$H_n\left(B_{2k+1}\left(\frac{x+1}{2}\right)\right) = (-1)^{\binom{n+1}{2}}\left(\frac{x}{2}\right)^{n+1}\prod_{\ell=1}^{n}\left(\frac{\ell^4(x^2-\ell^2)}{4(2\ell+1)(2\ell-1)}\right)^{n+1-\ell}.$$

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Sequence a_k, Orthogonal Polynomials $P_n(y)$ with

$$P_{n+1}(y) = (y+\alpha_n)P_n(y) - \beta_n P_{n-1}(y)$$

1. $H_n(a_k) = a_0^{n+1}\beta_1^n \cdots \beta_n$.
2. $$\sum_{k=0}^{\infty} a_k z^k = \cfrac{a_0}{1+\alpha_0 z - \cfrac{\beta_1 z^2}{1+\alpha_1 z - \cfrac{\beta_2 z^2}{1+\alpha_2 z - \ddots}}}$$

Sequences

$$B_{2k+1}\left(\frac{x+1}{2}\right), E_{2k}\left(\frac{x+1}{2}\right), E_{2k+1}\left(\frac{x+1}{2}\right), E_{2k+2}\left(\frac{x+1}{2}\right),$$

$$B_k\left(\frac{x+r}{q}\right) - B_k\left(\frac{x+s}{q}\right), E_k\left(\frac{x+r}{q}\right) \pm E_k\left(\frac{x+s}{q}\right),$$

$$kE_{k-1}(x), B_{k+1,x_{8,1}}(x), B_{k+1,x_{8,2}}(x), B_{k+1,x_{12,1}}(x), B_{k+1,x_{12,2}}(x),$$

$$(2k+1)E_{2k}, (2^{2k+2}-1)B_{2k+2}, (2k+1)B_{2k}\left(\frac{1}{2}\right), (2k+3)B_{2k+2},$$

$a_k,\ k \geq 1$	B_{k-1}	B_{2k}	$(2k+1)B_{2k}$	$(2^{2k}-1)B_{2k}$
a_0	0	1	1	0

$a_k,\ k \geq 1$	E_{2k-2}	$E_{k-1}(1)$	$E_{k+3}(1)$	$E_{2k-1}(1)$	$E_{2k+5}(1)$	$\frac{E_k(1)}{k!}$
a_0	0	0	$-\frac{1}{4}$	0	$\frac{1}{2}$	1

$a_k,\ k \geq 1$	$\frac{E_{2k-1}(1)}{(2k-1)!}$	$E_{2k-2}\left(\frac{x+1}{2}\right)$	$(2k+1)E_{2k}$		
a_0	0	0	0		

corresponding proofs. However, due to the close connection between Hankel determinants and orthogonal polynomials, especially as given by (5.11), it makes sense to use the latter forms as standard format. The following identities may serve to easily pass from one form to the other:

$$\prod_{\ell=1}^{n} \ell! = \prod_{\ell=1}^{n} \ell^{n+1-\ell}; \tag{7.5}$$

$$\prod_{\ell=1}^{n}(2\ell+\nu)! = \nu!^n \prod_{\ell=1}^{n}((2\ell-1+\nu)(2\ell+\nu))^{n+1-\ell}, \quad \nu = 0,1,2; \tag{7.6}$$

$$\prod_{\ell=1}^{n}(4\ell+\nu)! = \nu!^n \prod_{\ell=1}^{n}((4\ell-3+\nu)\cdots(4\ell+\nu))^{n+1-\ell}, \quad \nu = 0,1,2,3. \tag{7.7}$$

These identities, which actually hold in greater generality, can be verified without much difficulty.

We are now ready to list the identities for Hankel determinants, mostly given in a standard format and organized in a couple of tables. The references provided are not necessarily the first occurrences in the literature.

7.1. *Identities with nonzero terms for all n*

Most identities have nonzero Hankel determinants for all positive integers n; we present them in the format

$$H_n(b_k) = (-1)^{\varepsilon(n)} \cdot a^{n+1} \cdot \prod_{\ell=1}^{n} b(\ell)^{n-1-\ell}. \tag{7.8}$$

Here, the column for $\varepsilon(n)$ could be eliminated by incorporating a $-$ sign in a or $b(\ell)$, as appropriate. However, we decided to make the sign pattern more explicit.

b_k	$\varepsilon(n)$	a	$b(\ell)$	Reference
B_k	$\binom{n+1}{2}$	1	$\frac{\ell^4}{4(2\ell+1)(2\ell-1)}$	[14, (3.56)]
B_{k+1}	$\binom{n+2}{2}$	$\frac{1}{2}$	$\frac{\ell^2(\ell+1)^2}{4(2\ell+1)^2}$	[14, (3.57)]
B_{k+2}	$\binom{n+1}{2}$	$\frac{1}{6}$	$\frac{\ell(\ell+1)^2(\ell+2)}{4(2\ell+1)(2\ell+3)}$	[14, (2.38)]
B_{2k+2}	0	$\frac{1}{6}$	$\frac{\ell^3(\ell+1)(2\ell-1)(2\ell+1)^3}{(4\ell-1)(4\ell-1)^2(4\ell+3)}$	[14, (3.59)]
B_{2k+4}	$n+1$	$\frac{1}{30}$	$\frac{\ell(\ell+1)^3(2\ell+1)^3(2\ell+3)}{(4\ell+1)(4\ell-3)^2(4\ell+5)}$	[14, (3.60)]

(*Continued*)

b_k	$\varepsilon(n)$	a	$b(\ell)$	Reference
$B_{2k}\left(\frac{1}{2}\right)$	0	1	$\frac{\ell^4(2\ell-1)^4}{(4\ell-3)(4\ell-1)^2(4\ell+1)}$	[4, (41)]
$(2^{2k+2}-1)B_{2k+2}$	0	$\frac{1}{2}$	$\ell^3(\ell+1)$	Corollary 5.3
$(2k+1)B_{2k}\left(\frac{1}{2}\right)$	0	1	$\frac{\ell^6}{4(2\ell+1)(2\ell-1)}$	Corollary 5.4
$(2k+3)B_{2k+2}$	0	$\frac{1}{2}$	$\frac{\ell^3(\ell+1)^3}{4(2\ell+1)^2}$	Corollary 6.3
$B_{2k+1}\left(\frac{x+1}{2}\right)$	$\binom{n-1}{2}$	$\frac{x}{2}$	$\frac{\ell^4(x^2-\ell^2)}{4(2\ell+1)(2\ell-1)}$	[7, Theorem 1.1]
E_k	$\binom{n-1}{2}$	1	ℓ^2	[1, (4.2)]
$E_k(x)$	$\binom{n-1}{2}$	1	$\frac{\ell^2}{4}$	[1, (5.2)]
$E_{k+1}(1)$	$\binom{n-1}{2}$	$\frac{1}{2}$	$\frac{\ell(\ell+1)}{4}$	[10, (H4)]
E_{2k}	0	1	$(2\ell-1)^2(2\ell)^2$	[14, (3.52)]
$E_{2k-1}(1)$	0	$\frac{1}{2}$	$\frac{\ell^2(2\ell-1)(2\ell+1)}{4}$	[16, (4.56)]
E_{2k-2}	$n+1$	1	$(2\ell)^2(2\ell+1)^2$	[14, (3.53)]
$E_{2k-3}(1)$	$n+1$	$\frac{1}{4}$	$\frac{\ell(\ell+1)(2\ell+1)^2}{4}$	[16, (4.57)]
$(2k+1)E_{2k}$	0	1	$(2\ell)^4$	Corollary 5.2
$(2k+2)E_{2k+1}(1)$	0	1	$\ell^3(\ell+1)$	Corollary 5.3
$\frac{E_{k+1}(1)}{(k+1)!}$	$\binom{n-1}{2}$	$\frac{1}{2}$	$\frac{1}{4(2\ell-1)(2\ell+1)}$	[10, (H12)]
$\frac{E_{2k+1}(1)}{(2k+1)!}$	0	$\frac{1}{2}$	$\frac{1}{16(4\ell-3)(4\ell-1)^2(4\ell+1)}$	[10, (H13)]
$\frac{E_{2k+3}(1)}{(2k+3)!}$	$n+1$	$\frac{1}{24}$	$\frac{1}{16(4\ell-1)(4\ell+1)^2(4\ell+3)}$	[10, (H22)]
$E_{2k}\left(\frac{x+1}{2}\right)$	$\binom{n-1}{2}$	1	$\frac{\ell^2}{4}(x^2-(2\ell-1)^2)$	[7, Corollary 5.2]

(*Continued*)

b_k	$\varepsilon(n)$	a	$b(\ell)$	Reference
$E_{2k+1}\left(\frac{x+1}{2}\right)$	$\binom{n-1}{2}$	$\frac{x}{2}$	$\frac{\ell^2}{4}(x^2-(2\ell)^2)$	[7, Corollary 5.2]
$E_{2k+2}\left(\frac{x+1}{2}\right)$	$\binom{n-1}{2}$	$\frac{x^2-1}{4}$	$\frac{\ell^2}{4}(x^2-(2\ell-1)^2)$	[7, Corollary 5.2]

7.2. *Identities with zero terms for all even n*

A second class of identities have zero Hankel determinants for all positive even integers, that is,

$$H_{2m}(b_k) = 0; \tag{7.9}$$

we then present the nonzero terms in the format

$$H_{2m+1}(b_k) = (-1)^{m-1} \cdot a^{2(m+1)} \cdot \prod_{\ell=1}^{m} b(\ell)^{2(m+1-\ell)}. \tag{7.10}$$

b_k	a	$b(\ell)$	Reference
E_{k+1}	1	$(2\ell)^2(2\ell-1)^2$	[10, (H8)]
$E_{k+2}(1)$	$\frac{1}{4}$	$\frac{\ell(\ell-1)(2\ell+1)^2}{4}$	[10, (H11)][a]
$(0, E_1(1), E_2(1), \ldots)$	$\frac{1}{2}$	$\frac{\ell^2(2\ell-1)(2\ell+1)}{4}$	[10, (H9)]
$kE_{k-1}(x)$	1	ℓ^4	(3.8)
kE_{k-1}	1	$(2\ell)^4$	(3.9)
$\left(0, \frac{E_1(1)}{1!}, \frac{E_2(1)}{2!}, \ldots\right)$	$\frac{1}{2}$	$\frac{1}{16(4\ell-3)(4\ell-1)^2(4\ell-1)}$	[10, (H15)]
$\frac{E_{k+2}(1)}{(k+2)!}$	$\frac{1}{24}$	$\frac{1}{16(4\ell-1)(4\ell+1)^2(4\ell-3)}$	[10, (H14)]
$B_k\left(\frac{x+r}{q}\right) - B_k\left(\frac{x+s}{q}\right)$	$\frac{s-r}{q^{m+1}}$	$\frac{\ell^4((s-r)^2-(q\ell)^2)}{4(2\ell-1)(2\ell+1)}$	(3.5)
$E_k\left(\frac{x+r}{q}\right) - E_k\left(\frac{x+s}{q}\right)$	$\frac{s-r}{q}$	$\frac{\ell^2((s-r)^2-(2q\ell)^2)}{4q^2}$	(4.4)

[a] It appears that the author intended the determinant to be H_{2n+1}.

7.3. *Miscellaneous identities*

We now collect a number of identities that do not fit into Sec. 7.1 or 7.2. We begin with a few that are, however, closely related to some identities in the two tables above. The first of these identities was adapted from Andrews and Wimp [2, p. 441]:

$$H_n\left(\frac{B_k}{k!}\right) = (-1)^{\binom{n+1}{2}}(n+1)\prod_{\ell=1}^{n}\left(\frac{1}{4(2\ell-1)(2\ell+1)}\right)^{n-1-\ell}. \tag{7.11}$$

The next three identities are due to Krattenthaler and were published in [8, p. 346].

$$H_n\left(\frac{B_{2k+2}}{(2k+2)!}\right) = \left(\frac{1}{4}\right)^{(n+1)^2}\prod_{\ell=1}^{2n+1}\left(\frac{1}{2\ell+1}\right)^{2n+2-\ell}, \tag{7.12}$$

$$H_n\left(\frac{B_{2k+4}}{(2k+4)!}\right) = \left(\frac{-1}{36}\right)^{n-1}\left(\frac{1}{4}\right)^{(n+1)^2}\prod_{\ell=1}^{2n+1}\left(\frac{1}{2\ell+3}\right)^{2n+2-\ell}, \tag{7.13}$$

$$H_n\left(\frac{B_{2k+6}}{(2k+6)!}\right) = \frac{(n+2)(2n+5)}{3\cdot 60^{2n-2}}\left(\frac{1}{4}\right)^{(n+1)^2}\prod_{\ell=1}^{2n+1}\left(\frac{1}{2\ell+5}\right)^{2n+2-\ell}. \tag{7.14}$$

The identity (7.14) was slightly changed from its original form. The following three identities were adapted from (H21), (H23), and (H24), respectively, in [10].

$$H_n\left(\frac{E_{k+3}(1)}{(k+3)!}\right) = (-1)^{\binom{n+1}{2}}\left(\frac{1}{24}\right)^{n+1}\prod_{\ell=1}^{n}\left(\frac{1}{4(2\ell+1)(2\ell+3)}\right)^{n+1-\ell} \times \begin{cases}\binom{n+3}{2}, & (n \text{ odd}),\\ \binom{n+2}{2}, & (n \text{ even});\end{cases} \tag{7.15}$$

$$H_n\left(\frac{E_{2k+5}(1)}{(2k+5)!}\right) = \left(\frac{1}{2\cdot 6!}\right)^{n+1}\binom{2n+4}{2} \times \prod_{\ell=1}^{n}\left(\frac{1}{16(4\ell+1)(4\ell+3)^2(4\ell+5)}\right)^{n+1-\ell}; \tag{7.16}$$

$$H_n\left(\frac{E_{2k+7}(1)}{(2k+7)!}\right) = \left(\frac{-1}{5\cdot 8!}\right)^{n+1}\frac{4n^2+18n+17}{3}\binom{2n+6}{4} \times \prod_{\ell=1}^{n}\left(\frac{1}{16(4\ell+3)(4\ell+5)^2(4\ell+7)}\right)^{n+1-\ell}. \tag{7.17}$$

Another identity of a similar nature is

$$H_n(B_{k+2}(-1)) = (-1)^{\binom{n+1}{2}}\left(\frac{1}{6}\right)^{n+1}((n+1)(n+2)^2(n+3)+1) \times \prod_{\ell=1}^{n}\left(\frac{\ell(\ell+1)^2(\ell+2)}{4(2\ell+1)(2\ell+3)}\right)^{n+1-\ell}, \tag{7.18}$$

which can be found in [9, Eq. (7.2)], in a slightly different form. In this connection Fulmek and Krattenthaler also showed that

$$H_n(B_k - 2B_{k+1} + B_{k+2}) = H_n(B_{k+2}(-1)) \quad (n \geq 0).$$

Furthermore, they derived identities for

$$H_{2m}\left(B_{k+2}\left(-\frac{1}{2}\right)\right) \quad \text{and} \quad H_{2m+1}\left(B_{k+2}\left(-\frac{1}{2}\right)\right),$$

involving ${}_4F_3$ hypergeometric functions [9, Eqs. (7.3), (7.4)] and identities for

$$H_{2m}\left(B_{k+2}\left(\frac{1}{2}\right)\right) \quad \text{and} \quad H_{2m+1}\left(B_{k+2}\left(\frac{1}{2}\right)\right),$$

which involve certain finite sums [9, Eqs. (7.5), (7.6)]. At this point we also mention the identities for

$$H_{2m}\left(E_k\left(\frac{x+r}{q}\right)+E_k\left(\frac{x+s}{q}\right)\right) \quad \text{and} \quad H_{2m+1}\left(E_k\left(\frac{x-r}{q}\right)+E_k\left(\frac{x-s}{q}\right)\right),$$

which were obtained in (4.5) and (4.6) above.

We conclude this list of identities with a very general formula, which is also due to Fulmek and Krattenthaler [9, Eq. (5.13)]. Here a well-known symbolic notation is used, where after expansion each power $\mathcal{B}^j$ is replaced by the Bernoulli number B_j. Also, the shifted factorial $(a)_j$ is defined by $(a)_j := a(a+1)\cdots(a+j-1)$ for $j \geq 1$, and $(a)_0 = 1$. Thus, for example, we have

$$\mathcal{B}^3(\mathcal{B}+1)_2 = \mathcal{B}^3(\mathcal{B}+1)(\mathcal{B}-2) = \mathcal{B}^5 + 3\mathcal{B}^4 + 2\mathcal{B}^3 \equiv B_5 + 3B_4 - 2B_3.$$

We can now state the identity in question, which is, in fact, again of the form (7.8).

For integers $a, b \geq 1$ and $c, d \geq 0$, we have

$$H_n(\mathcal{B}^{k+2}(\mathcal{B}+1)_{a-1}(\mathcal{B}+1)_{b-1}(-\mathcal{B}+1)_{c-1}(-\mathcal{B}+1)_{d-1}) = (-1)^{\binom{n+1}{2}}\left(\frac{(a+c-1)!(b+c-1)!(a+d-1)!(b+d-1)!}{(a+b+c+d-1)!}\right)^{n+1} \times \prod_{\ell=1}^{n}\left(\frac{\ell(a+c+\ell-1)(b+c+\ell-1)(a+d+\ell-1)}{(a+b+c+d-2\ell-3)(a+b+c+d+2\ell-2)^2} \times \frac{(b+d+\ell-1)(a+b+c+d+\ell-2)}{(a+b+c+d+2\ell-1)}\right)^{n+1-\ell}, \tag{7.19}$$

where in the case $c = 0$ or $d = 0$ we interpret $(-\mathcal{B}+1)_{-1}$ as $1/(-\mathcal{B})$.

As mentioned in [9, p. 626], the cases $a = b = 1, c = d = 0$ and $a = b = c = d = 1$ give the first and third entries, respectively, in the table in Sec. 7.1. Similarly, $a = b = c = 1, d = 0$, would give the second entry in this table.

7.4. *Other related sequences*

We made the conscious decision to restrict ourselves to Bernoulli and Euler numbers and polynomials in Secs. 7.1–7.3. We conclude this section with a few remarks on related sequences.

(1) While number theorists and researchers in special functions tend to favor the definition (1.2) for Euler numbers, combinatorists typically prefer the alternative sequence $\mathbf{E}_n$ defined by

$$\tan t + \sec t = \sum_{n=0}^{\infty} \mathbf{E}_n \frac{t^n}{n!}, \tag{7.20}$$

where we use a different font to avoid possible confusion. The first few terms, starting with $\mathbf{E}_0$, are 1, 1, 1, 2, 5, 16, 61, 272, 1385; they are all positive integers. By comparing the generating function (7.20) with (1.2), it is not difficult to see that, for all $k \geq 0$,

$$\mathbf{E}_{2k} = (-1)^k E_{2k}, \tag{7.21}$$

$$\mathbf{E}_{2k+1} = (-1)^k 2^{2k+1} E_{2k-1}(1). \tag{7.22}$$

Using these identities and (2.14), any Hankel determinant identity for the sequences on the right immediately gives identities for the ones on the left, and vice versa; in fact, this is how we imported the numerous identities from Han's recent paper [10]. There are eight more identities in [10] for the "mixed" sequence $(\mathbf{E}_k)$, namely for

$$H_n(\mathbf{E}_{k+\mu}), \quad \mu = 0,1,2, \quad \text{and} \quad H_n(\mathbf{E}_{k+\nu}/(k+\nu)!), \quad \nu = 0,1,2,3,4.$$

(2) By (7.20) it is clear that the numbers $\mathbf{E}_{2k-1}$, $k \geq 1$, are the same as the *tangent numbers* (or *tangent coefficients*) T_k, which are also known to have the form

$$T_k = (-1)^{k-1} 2^{k} \frac{2^{2k}-1}{2k} B_{2k}$$

(see, e.g., [18, Eq. 4.19.3]). By (7.22) we have $T_k = (-1)^{k-1}2^{2k-1}E_{2k-1}(1)$; therefore all identities for $E_k(1)$ can also be seen as identities for tangent numbers.

(3) Euler numbers of an integer order $p \geq 1$ are defined by a generating function that is the pth power of the left-hand identity in (1.2). Already Al-Salam and Carlitz [1] found the Hankel determinants of the sequence of these higher-order Euler numbers. More recently, Han [10] dealt with other related sequences of higher-order Euler numbers, and the second author and Shi [12] determined the orthogonal polynomials of higher-order Euler polynomials, which also led

Theorem (L. J and Y. Li)

(1.4)

$$H_n\left(\frac{B_{2k+1}\left(\frac{x+1}{2}\right)}{2k+1}\right) = \left(\frac{x}{2}\right)^{n+1} \prod_{\ell=1}^{n} \left(\frac{(2\ell)^2(2\ell-1)^2(x^2-(2\ell-1)^2)(x^2-(2\ell)^2)}{16(4\ell-3)(4\ell-1)^2(4\ell+1)}\right)^{n+1-\ell}.$$

(1.5)

$$H_n\left(\frac{B_{2k+1}\left(\frac{x+1}{2}\right)}{2k+1}\right) = \left(\frac{x^3-x}{24}\right)^{n+1} \times \prod_{\ell=1}^{n} \left(\frac{(2\ell)^2(2\ell+1)^2(x^2-(2\ell+1)^2)(x^2-(2\ell)^2)}{16(4\ell-1)(4\ell+1)^2(4\ell+3)}\right)^{n+1-\ell}.$$

(1.2)

$$H_n\left(\frac{B_{2k+5}\left(\frac{x+1}{2}\right)}{2k+5}\right) = \frac{1}{5\cdot 2^{n+2}} \prod_{i=1}^{n} \frac{(2i+3)!^2(2i+2)!^2}{(4i+5)!(4i+4)!} \prod_{\ell=0}^{n} (x-2n-1+2\ell)_{4n-4\ell+3} \times \sum_{i=1}^{n+2} \frac{(2i-1)\left(n+\frac{5}{2}\right)_{i-1}\left(\frac{x}{2}+\frac{1}{2}\right)_{n+2}\left(\frac{x}{2}-n-\frac{3}{2}\right)_{n+2}}{\left(n-i+\frac{5}{2}\right)_i (n+2-i)!(n+1+i)!\,(x^2-(2i-1)^2)},$$

and

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$$H_n\left(\frac{B_{2k+5}\left(\frac{x+1}{2}\right)}{2k+5}\right) = \frac{1}{5\cdot 2^{n+2}}\prod_{i=1}^{n}\frac{(2i+3)!^2(2i+2)!^2}{(4i+5)!(4i+4)!}\prod_{\ell=0}^{n}(x-2n-1+2\ell)_{4n-4\ell+3} \times \sum_{i=1}^{n+2}\frac{(2i-1)\left(n+\frac{5}{2}\right)_{i-1}\left(\frac{x}{2}+\frac{1}{2}\right)_{n+2}\left(\frac{x}{2}-n-\frac{3}{2}\right)_{n+2}}{\left(n-i+\frac{5}{2}\right)_i(n+2-i)!(n+1+i)!\,(x^2-(2i-1)^2)},$$

and

metric regression. Define

$$I_k := \sum_{c=1}^{r} c^k$$

and the Hankel matrix

$$V_n := \begin{pmatrix} I_0 & I_2 & \cdots & I_{2n} \\ I_2 & I_4 & \cdots & I_{2n+2} \\ \vdots & \vdots & \ddots & \vdots \\ I_{2n} & I_{2n+2} & \cdots & I_{4n} \end{pmatrix}.$$

q-riosity

q-riosity

Al-Salam [1] defined two q-analogs of the Bernoulli polynomials by

$$\frac{te_q(tx)}{e_q(t)-1}=\sum_{n=0}^{\infty}\frac{t^n}{[n]!}B_{n,q}(x),\qquad \frac{tE_q(tx)}{E_q(t)-1}=\sum_{n=0}^{\infty}\frac{t^n}{[n]!}\beta_{n,q}(x),$$

where

$$e_q(x)=\frac{1}{(x(1-q);q)_\infty},\qquad E_q(x)=(-x(1-q);q)_\infty.$$

Al-Salam pointed out that $\beta_{n,q}(x)$ is essentially $B_{n,q}(x)$ with q replaced by $1/q$. It is clear that $e_q(x)E_q(-x)=1$ for all $x\in\mathbb{C}$. The functions $E_q(x)$ and $e_q(x)$ have the series representation

$$e_q(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma_q(k+1)};\quad |x|<1,\quad \text{and}\quad E_q(x)=\sum_{k=0}^{\infty}q^{\frac{k(k-1)}{2}}\frac{x^k}{\Gamma_q(k+1)};\ x\in\mathbb{C}.$$

Nalci and Pashaev in [29] introduced a q-analog of the Bernoulli polynomials by

$$\frac{te_q(tx)}{e_q(t/2)E_q(t/2)-1}=\sum_{n=0}^{\infty}b_n(x;q)\frac{t^n}{[n]!},\tag{1.4}$$

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$$q(\beta q+1)^m-\beta_m=\begin{cases}1, & m=1,\\ 0, & m>1.\end{cases}$$

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Al-Salam pointed out that $\beta_{n,q}(x)$ is essentially $B_{n,q}(x)$ with q replaced by $1/q$. It is clear that $e_q(x)E_q(-x) = 1$ for all $x \in \mathbb{C}$. The functions $E_q(x)$ and $e_q(x)$ have the series representation

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma_q(k+1)}; \quad |x| < 1, \quad \text{and} \quad E_q(x) = \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{x^k}{\Gamma_q(k+1)}; \; x \in \mathbb{C}.$$

Nalci and Pashaev in [29] introduced a q-analog of the Bernoulli polynomials by

$$\frac{te_q(tx)}{e_q(t/2)E_q(t/2)-1} = \sum_{n=0}^{\infty} b_n(x;q) \frac{t^n}{[n]!}, \tag{1.4}$$

$$q(\beta q+1)^m - \beta_m = \begin{cases} 1, & m = 1, \\ 0, & m > 1. \end{cases}$$

Théorème 4.2. *On a, pour les matrices d'indices* $0 \le i,j \le n-1$,

$$\det\left(\beta_{i+j}\right)_{i,j} = (-1)^{\binom{n}{2}} q^{\binom{n}{3}} \prod_{i=1}^{n-1} \frac{[i]!_q^6}{[2i]!_q[2i+1]!_q}, \tag{4.7}$$

$$\det\left(\beta_{i+j+1}\right)_{i,j} = \frac{(-1)^{\binom{n+1}{2}}}{[2]_q} q^{\binom{n+1}{3}} \prod_{i=1}^{n-1} \frac{[i]!_q^3[i+1]!_q^3}{[2i+1]!_q[2i+2]!_q}, \tag{4.8}$$

$$\det\left(\beta_{i+j+2}\right)_{i,j} = \frac{(-1)^{\binom{n}{2}}}{[2]_q[3]_q} q^{\binom{n+2}{3}} \prod_{i=1}^{n-1} \frac{[i]!_q[i+1]!_q^4[i+2]!_q}{[2i+2]!_q[2i+3]!_q}, \tag{4.9}$$

$$\det\left(\beta_{i+j+3}\right)_{i,j} = \frac{(-1)^{\binom{n+1}{2}}}{[3]_q^2[4]_q} q^{\binom{n+2}{3}} \left(q^{\binom{n+2}{2}} + (-1)^n\right) \prod_{i=1}^{n-1} \frac{[i+1]!_q^3[i+2]!_q^3}{[2i+3]!_q[2i+4]!_q} \tag{4.10}$$

q-Euler numbers

Theorem (S. Chern and L. J)

Let

$$\epsilon_n := \frac{1}{(1-q)^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1+q}{1+q^{k+1}}.$$

Then,

$$\det_{0\le i,j\le n}(\epsilon_{i+j}) = \frac{(-1)^{\binom{n+1}{2}} q^{\frac{1}{4}\binom{2n+2}{3}}}{(1-q)^{n(n+1)}} \prod_{k=1}^{n} \frac{(q^2,q^2;q^2)_k}{(-q,-q^2,-q^2,-q^3;q^2)_k}$$

$$\det_{0\le i,j\le n}(\epsilon_{i+j+1}) = \frac{(-1)^{\binom{n+2}{2}} q^{\frac{1}{4}\binom{2n+4}{3}}}{(1-q)^{n(n+1)}(1+q^2)^{n+1}} \prod_{k=1}^{n} \frac{(q^2,q^4;q^2)_k}{(-q^2,-q^3,-q^3,-q^4;q^2)_k}$$

$$\det_{0\le i,j\le n}(\epsilon_{i+j+2}) = \frac{(-1)^{\binom{n+2}{2}} q^{\frac{1}{4}\binom{2n+4}{3}} (1+q)^n \left(1-(-1)^n q^{(n+2)^2}\right)}{(1-q)^{n(n+1)}(1+q^2)^{2(n+1)}(1+q^3)^{n+1}} \times \prod_{k=1}^{n} \frac{(q^4,q^4;q^2)_k}{(-q^3,-q^4,-q^4,-q^5;q^2)_k}.$$

$$\mathcal{J}_{\ell,n}(z) := {}_3\phi_2\left(\begin{matrix} q^{-n}, -q^{n+\ell+1}, z \\ q^{\ell+1}, 0 \end{matrix}; q, q\right).$$

$$\Phi\left(\begin{bmatrix} n+1, z \\ n \end{bmatrix}_q\right) = \frac{1+q}{q} - \frac{1}{q(-q^2;q)_n}.$$

$$\mathcal{J}_{\ell,n}(z) := {}_3\phi_2\left(\begin{matrix} q^{-n}, -q^{n+\ell+1}, z \\ q^{\ell+1}, 0 \end{matrix}; q, q\right).$$

$$\Phi\left(\begin{bmatrix} n+1, z \\ n \end{bmatrix}_q\right) = \frac{1+q}{q} - \frac{1}{q(-q^2;q)_n}.$$

$$\begin{bmatrix} m, z \\ n \end{bmatrix}_q := \frac{1}{[n]_q!} \prod_{k=m-n+1}^{m} ([k]_q + q^k z).$$

What is now/next?

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Theorem (S. Chern, L. J, and S. Li)

The leading coefficient of $H_n\left(B_{2k}\left(\frac{1+x}{2}\right)\right)$ *is*

$$H_n(2^k B_k).$$

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$$K_n(x) := \sum_{j=0}^{n} \binom{n}{j} B_{n+j}\left(\frac{x}{2}\right) \Rightarrow H_n(K_k(x)) = H_n\left(B_{2k}\left(\frac{x+1}{2}\right)\right).$$

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q-binomial?