Shuffle to One, Shuffle to Normal

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Acknowledgment

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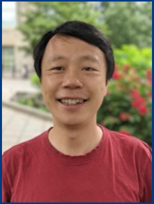

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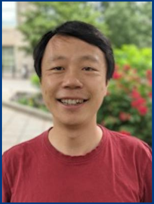

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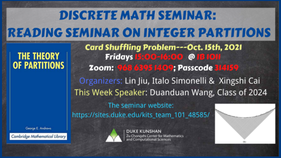
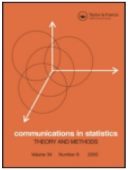


Duanduan Wang

Model



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A discrete probability problem in card shuffling

M. Bhaskara Rao, Haimeng Zhang, Chunfeng Huang & Fu-Chih Cheng

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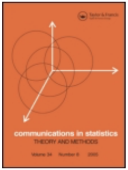

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"The original question raised was to determine how many times catalysts are expected to be added in order to get a single lump of all molecules."

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Shuffle to Normal

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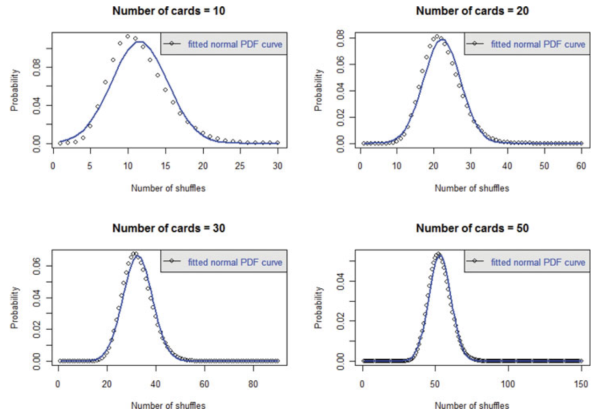
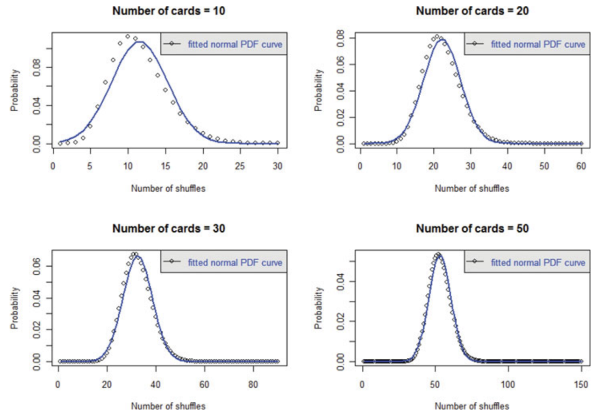


Figure 1. Shuffling distributions with normal curves fitted.

Shuffle to Normal



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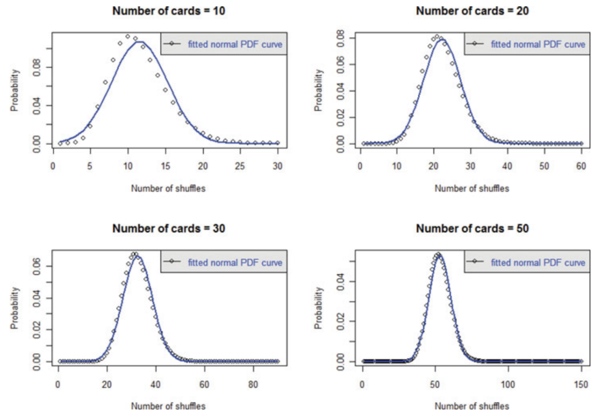

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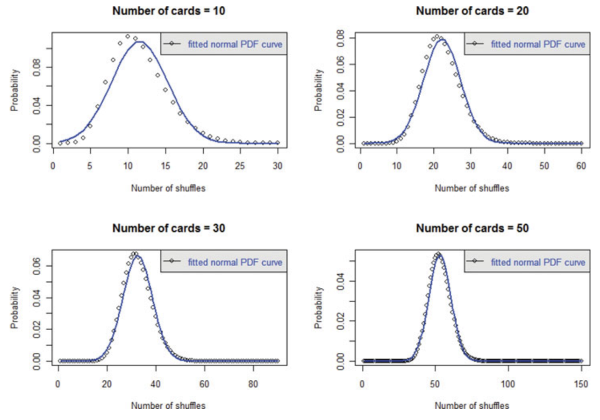

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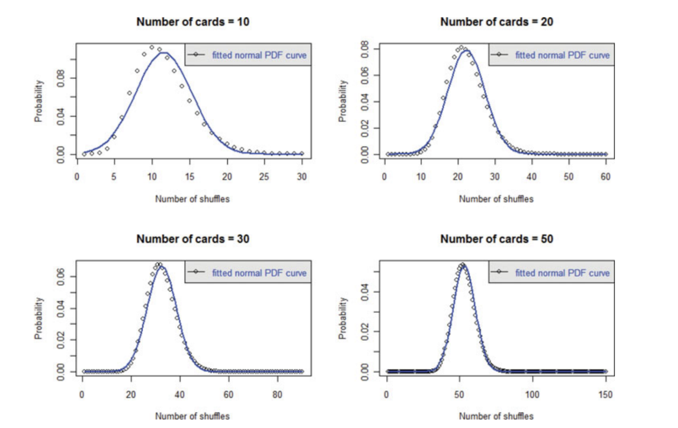

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Remark

Experiments shows $\mathbb{E}[X_n] - n \sim \log n$.

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  2. And if given the sequence $p_{n,k}$, to find the asymptotic expression of μ_n can be considered independent of the shuffling model.

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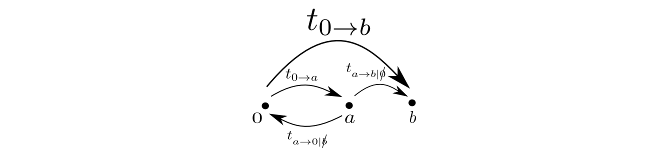

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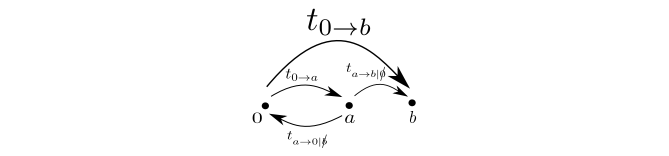

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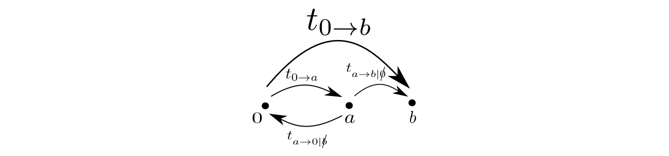

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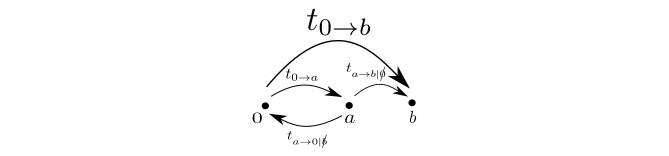

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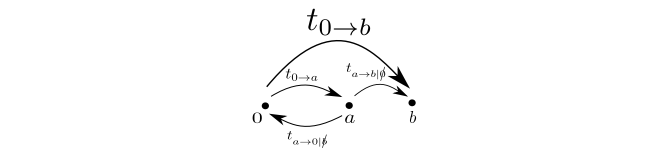

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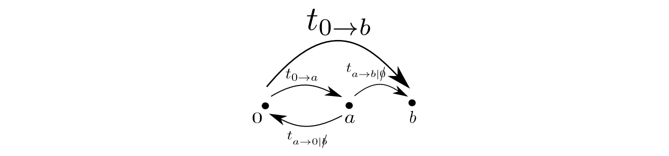

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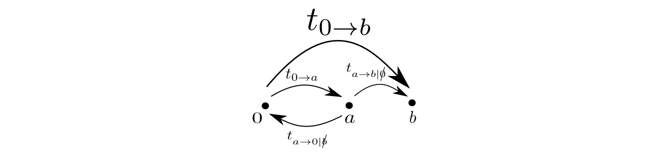

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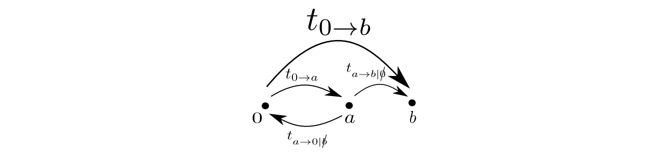

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$p_{n,k} = A(n, k)/n!$

Definition

A permutation of n integers $1, 2, ..., n$ is said to have k lumps if and only if when the numbers are read from left to right, after the numbers in consecutive increasing order are merged, there are exactly k mergers including those standing alone.

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2. $A(n,k)$: A010027.

```
EXAMPLE        Triangle starts:
                 1;
                 1, 1;
                 1, 2,   3;
                 1, 3,   9,  11;
                 1, 4,  18,  44,   53;
                 1, 5,  30, 110,  265,   309;
                 1, 6,  45, 220,  795,  1854,   2119;
                 1, 7,  63, 385, 1855,  6489,  14833,  16687;
                 1, 8,  84, 616, 3710, 17304,  59332, 133496,  148329;
                 1, 9, 108, 924, 6678, 38934, 177996, 600732, 1334961, 1468457;
                 ...
               For n=3, the permutations 123, 132, 213, 231, 312, 321 have respectively
                 2,0,0,1,1,0 consecutive ascending pairs, so row 3 of the triangle is
                 3,2,1. - N. J. A. Sloane, Apr 12 2014
               In the alternative definition, T(4,2)=3 because we have 234.1, 4.123, and
                 34.12 (the blocks are separated by dots). - Emeric Deutsch, May 16 2010
```

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Lemma ($U(n) := \sum_{k=1}^{n} u_k$)

$$\sum_{n=1}^{N} u_n v_n = U(N) v_{N+1} + \sum_{n=1}^{N} U(n)\left(v_n - v_{n+1}\right)$$

$\mathrm{Var}[X_n]$

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Theorem (M. Rao, H. Zhang, C. Huang, and F.-C. Cheng)

For any $n \geq 2$,

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1. The ultimate goal is to show

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Let $\{\lambda_n\} \subset \mathbb{C}$ *with* $\lambda_n \sim Mn^L$ *as* $n \to \infty$ *for fixed* $L \in \mathbb{N} \cup \{0\}$ *and* $M \in \mathbb{C}$*, where* $M \neq 0$ *if* $L \neq 0$*. Define sequence* ξ_n *by the recurrence*

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$$\sum_{t=1}^{n}S_n(t)\cdot\frac{1}{t^2}=\frac{1}{n^2}+\sum_{m=1}^{n-1}\frac{(n-m-1)!}{n\cdot n!}+\sum_{m=1}^{n-1}\frac{(n-m)!}{m\cdot n!}\left(\mathcal{H}_n-\mathcal{H}_{n-m}\right)$$

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Theorem (S. Chern, L. Jiu, and I. Simonelli)

Let $\ell \in \mathbb{N} \cup \{0\}$, *then for* $n \geq \ell$,

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$$B_\ell = \sum_{m=0}^{\ell} \left\{ \begin{matrix} \ell \\ m \end{matrix} \right\},$$

where $\left\{ \begin{matrix} \ell \\ m \end{matrix} \right\}$ *is the Stirling numbers of the second kind.*

Theorem (S. Chern, L. Jiu, and I. Simonelli)

Let $\ell \in \mathbb{N} \cup \{0\}$, *then for* $n \geq \ell$,

$$\sum_{k=1}^{n} \frac{A(n,k)}{n!}(n-k)^\ell = B_\ell - (B_{\ell+1} - B_\ell) \cdot \frac{1}{n}.$$

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     with $a_1 = 1$ and $b_1 = 0$. Apparently, $b_n = \mu_n$ and it seems that a_n has a limit.

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A Galton–Watson tree $\mathcal{T}$ is a tree in which each node is given a random number of child nodes, where the numbers of child nodes are drawn independently from the same distribution ξ which is often called the offspring distribution.

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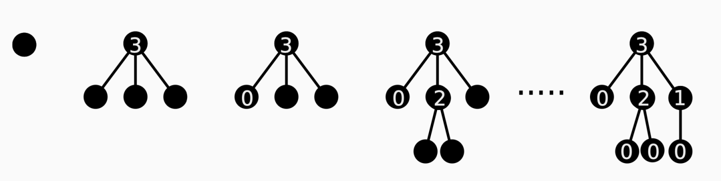

End: Any Questions?