# Hankel Determinants and Big $q$-Jacobi Polynomials for $q$-Euler Numbers

Lin Jiu

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DUKE KUNSHAN
Zu Chongzhi Center for Mathematics and Computational Sciences

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August 1st, 2025


Dr. Shane Chern


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- The big $q$-Jacobi polynomials

$$\mathscr{J}_{\ell,n}(z) \quad := \quad {}_3\phi_2\left(\begin{matrix} q^{-n}, -q^{n+\ell+1}, z \\ q^{\ell+1}, 0 \end{matrix} ; q, q\right)$$


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## Theorem

$H_n(\mathrm{C}) = 1$ *for all* $n = 0, 1, \ldots$

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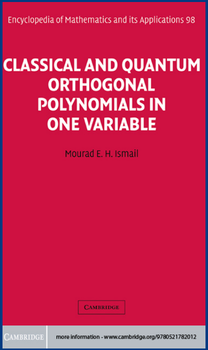

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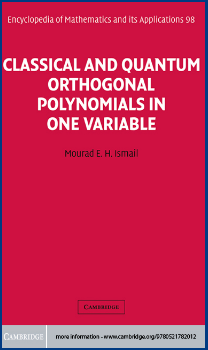

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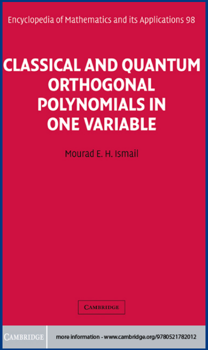

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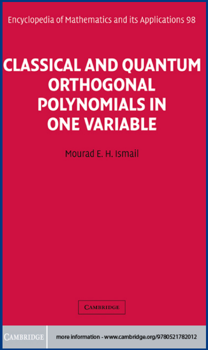


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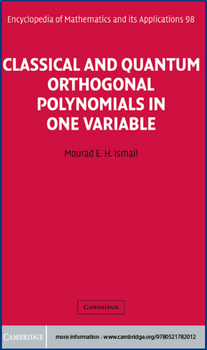


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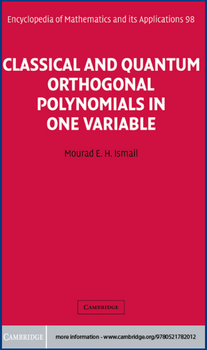


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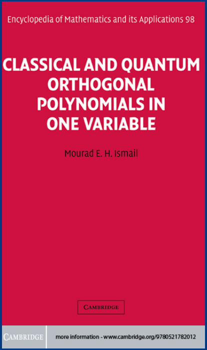


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$$P_{n+1} = (y+s_n)P_n(y) - t_nP_{n-1}(y) \Rightarrow \begin{cases} \sum_{n=0}^{\infty} c_n z^n = \cfrac{c_0}{1+s_0z-\cfrac{t_1z^2}{1+s_1z-\cfrac{t_2z^2}{\ddots}}} \\ H_n(\mathrm{c}) = c_0^{n+1}t_1^n t_2^{n-1}\cdots t_n \end{cases}$$

# Early Work with Karl Dilcher

K. Dilcher and L. Jiu

- Hankel determinants of shifted sequences of Bernoulli and Euler numbers, Contrib. Discrete Math. 18 (2023), 146–175.
- Hankel Determinants of sequences related to Bernoulli and Euler Polynomials, Int. J. Number Theory 18 (2022), 331–359.
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We computed the Hankel determinants of the following sequences:

$$B_{2n+1}\left(\tfrac{x+1}{2}\right) \qquad E_{2k}\left(\tfrac{x+1}{2}\right)$$
$$E_{2k+1}\left(\tfrac{x+1}{2}\right) \qquad E_{2k+2}\left(\tfrac{x+1}{2}\right)$$
$$B_k\left(\frac{x+r}{q}\right) \pm B_k\left(\frac{x+s}{q}\right) \qquad kE_{k-1}(x)$$
$$B_{k,\chi_q} \qquad (q=3,4,6)$$
$$\frac{B_{k,\chi_{2q,\ell}}}{k+1} \qquad (q=3,4;\ell=1,2)$$
$$E_k\left(\frac{x+r}{q}\right) \pm E_k\left(\frac{x+s}{q}\right) \qquad (2k+1)E_{2k}$$
$$(2^{2k+2}-1)B_{2k+2} \qquad (2k+1)B_{2k}(\tfrac{1}{2})$$
$$(2k+3)B_{2k} \qquad (2k+2)E_{2k+1}(1)$$

| $b_k, k \geq 1$ | $b_0$ | Prop. | $b_k, k \geq 1$ | $b_0$ | Prop. |
|---|---|---|---|---|---|
| $B_{k-1}$ | 0 | 3.1 | $E_{k+3}(1)$ | $(\frac{-1}{4})$ | 5.2 |
| $B_{2k}$ | (1) | 6.1 | $E_{2k-1}(1)$ | 0 | 3.3 |
| $(2k+1)B_{2k}$ | (1) | 6.2 | $E_{2k+5}(1)$ | $(\frac{1}{2})$ | 5.1 |
| $(2^{2k}-1)B_{2k}$ | (0) | 3.4 | $E_k(1)/k!$ | (1) | 3.6 |
| $(2k+1)E_{2k}$ | 0 | 3.5 | $E_{2k-1}(1)/(2k-1)!$ | 0 | 6.3 |
| $E_{2k-2}$ | 0 | 7.3 | $E_{2k-2}(\frac{x+1}{2})$ | 0 | 7.2 |
| $E_{k-1}(1)$ | 0 | 3.2 | $E_{k-1}^{(p)}$ | $\alpha \in \mathbb{R}$ | 8.1 |

TABLE 2. Summary of results.

# Bernoulli and Euler Polynomials

**Definition**

The *Bernoulli polynomials* $B_n(x)$ and *Euler polynomials* $E_n(x)$ are given by their exponential generating functions

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!} \quad \text{and} \quad \frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

Specific evaluations give *Bernoulli numbers* $B_n = B_n(0)$ and *Euler numbers* $E_n = 2^n E_n(1/2)$.

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## Theorem (Al-Salam and Carlitz)

$$H_n(B_k) = (-1)^{\binom{n+1}{2}} \prod_{k=1}^{n} \frac{(k!)^6}{(2k)!(2k+1)!} \quad \textit{and} \quad H_n(E_k) = (-1)^{\binom{n+1}{2}} \prod_{k=1}^{n} (k!)^2.$$

# $q$-analog

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The $q$-Bernoulli numbers were introduced by Carlitz as

$$\beta_n := \frac{1}{(1-q)^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{k+1}{[k+1]_q},$$

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## Conjecture (L. J)

$$H_n(\beta_k) = (-1)^{\binom{n+1}{2}} q^{\frac{(n-1)n(n+1)}{6}} \frac{\prod_{k=1}^{n} [k]_q^{6(n+1-k)}}{\prod_{k=1}^{2n+1} [k]_q^{2n+2-k}}.$$

# $q$-Bernoulli

LJ **Lin Jiu, Ph.D.**

RE: q-Bernoulli

To: Karl Dilcher, Shane Chern

February 18, 2023 at 11:12 PM

Good morning, Karl and Shane,

Admittedly, the expression can (or maybe not) be further simplified for the common powers of 1-q, the current expression looks good. I only have Mathematica code rather than Maple (as DKU does not support a Maple license); so I am not sending you the code. At least, the expression holds for n=0,1,\ldots,10.

Anyway, the paper Karl sent include the generating function of \beta_m, so probably, we can find its continued fraction expression; or maybe there are some other ways to prove it.

This could be a good starting point for some q-analogues.

Have a nice weekend,
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**See More** from Karl Dilcher

THE STARTING POINT

1. The Carlitz $\beta$

Carlitz [1, 2] generated the Bernoulli numbers to the sequence $\beta_n$, by the recurrence:

$$\sum_{k=0}^{n}\binom{n}{k}\beta_k q^{k+1}-\beta_m=\begin{cases}1, & m=1;\\ 0, & m>1,\end{cases}$$

with also the value $\beta_0=1$.

**Definition 1.** The $q$-bracket is defined by

$$[x]_q:=\frac{1-q^x}{1-q},$$

for all $x\in\mathbb{R}$ and $q>0$. The $q$-factorial is then defined by

$$[k]_q!:=[k]_q[k-1]_q\cdots[1]_q.$$

**Conjecture 2.**

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## Theorem (F. Chapton and J. Zeng, 2017)

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- *F. Chapoton and J. Zeng, "Nombres de q-Bernoulli-Carlitzet fractions continues", J. Théor. Nombres Bordeaux 29 (2017), no. 2, pp. 347-368.*

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## Theorem (S. Chern and L. J.)

$$\det_{0\le i,j\le n} \left(\epsilon_{i+j}\right) = \frac{(-1)^{\binom{n+1}{2}} q^{\frac{1}{4}\binom{2n+2}{3}}}{(1-q)^{n(n+1)}} \prod_{k=1}^{n} \frac{\left(q^2, q^2; q^2\right)_k}{\left(-q, -q^2, -q^2, -q^3; q^2\right)_k}$$

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# Big $q$-Jacobi Polynomial

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The $q$-hypergeometric series ${}_{r+1}\phi_r$ is defined as

$$
{}_{r+1}\phi_r \left( \begin{matrix} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix}; q; z \right) := \sum_{n \geq 0} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_n \, z^n}{(q, b_1, b_2, \ldots, b_r; q)_n},
$$

# Big $q$-Jacobi Polynomial

## Definition

The $q$-hypergeometric series ${}_{r+1}\phi_r$ is defined as

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(A; q)_n := \prod_{k=1}^{n} (1 - Aq^{k-1}), \quad (A_1, A_2, \ldots, A_n, q) := \prod_{j=1}^{n} (A_j; q)_n
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\mathscr{J}_{\ell,n}(z) := {}_3\phi_2\left(\begin{matrix} q^{-n},-q^{n+\ell+1},z \\ q^{\ell+1},0 \end{matrix};q,q\right) = \sum_{n\geq 0} \frac{(q^{-n},-q^{n+\ell+1},z;q)_n}{(q,q^{\ell+1},0;q)_n} q^n
$$

# Linear Functional

# Linear Functional

## Definition

The linear functional $\Phi$ on $\mathbb{Q}(q)[z]$ is defined by

$$\Phi\left(\begin{bmatrix} m, z \\ n \end{bmatrix}_q\right) = \frac{(-1)^{n-m} q^{n-m}}{(-q^2; q)_n},$$

where

$$\left[\begin{array}{c} m, z \\ n \end{array}\right]_q := \frac{1}{[n]_q!} \prod_{k=m-n+1}^{m} \left([k]_q + q^k z\right).$$

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## Theorem (S. Chern and L. J)

$\Phi(z^n) = \epsilon_n$.

- $\mathrm{c} = (c_0, c_1, \ldots, c_n, \ldots)$
- Orthogonal polynomials $P_n$, w. r. t. c:

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$$A_{\ell,n}\mathscr{J}_{\ell,n+1}(z) = (A_{\ell,n} + B_{\ell,n} - 1 + z)\,\mathscr{J}_{\ell,n}(z) - B_{\ell,n}\mathscr{J}_{\ell,n-1}(z),$$

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$$\mathscr{P}_{\ell,n}(z) = \frac{(-1)^n}{q^n (1-q)^n} \widetilde{\mathscr{J}}_{\ell,n}\left(\left(q^2 - q\right) z + q\right)$$

with

$$\widetilde{\mathscr{J}}_{\ell,n}(z) := \frac{\left(q^{\ell+1}; q\right)_n}{\left(-q^{n+\ell+1}; q\right)_n} \mathscr{J}_{\ell,n}(z).$$

# Final Piece

**Theorem (S. Chern and L. J)**

$$\Phi\left(\mathscr{P}_{0,n}(z)\right) = \begin{cases} \epsilon_0, & n = 0; \\ 0, & n \geq 1. \end{cases}$$

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## Remark

If we define

$$\Theta_\ell\left(\begin{bmatrix} n, z \\ n \end{bmatrix}_q\right) := \frac{\left(q^{\ell+1}; q\right)_n}{(q, -q^{\ell+2}; q)_n}$$

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And the corresponding sequence is

$$\xi_{\ell,n} := \frac{q^{(\ell+1)n}(-q; q)_n}{\left(-q^{\ell+2}; q\right)_n}.$$

# The End

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Or......

# Binomial Transform

**Theorem**

*Given a sequence* $\mathrm{c} = (c_0, c_1, \ldots)$ *and defined the sequence of polynomials*

$$c_k(x) := \sum_{\ell=0}^{k} \binom{k}{\ell} c_\ell x^{k-\ell},$$

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$$H_n(c_k) = H_n(c_k(x)).$$

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## Problem

How about the $q$-binomial transform? Given a sequence $\alpha_n$, we now consider

$$\alpha_k(x) := \sum_{\ell=0}^{k} q^{\binom{\ell}{2}} \begin{bmatrix} k \\ \ell \end{bmatrix}_q \alpha_{k-\ell} x^\ell \quad \text{and} \quad \widetilde{\alpha}_k(x) := \sum_{\ell=0}^{k} q^{\binom{\ell}{2}} \begin{bmatrix} k \\ \ell \end{bmatrix}_q \alpha_\ell x^{k-\ell}$$

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## Theorem (S. Chern, L. J., S. Li, and L. Wang)

1. *For every* $n \geq 0$, $H_n(\alpha_k(x))$ *is a polynomial in* $x$ *of degree* $n(n+1)$ *with leading coefficient*

$$\left[x^{n(n+1)}\right] H_n\left(\alpha_k(x)\right) = \alpha_0^{n+1}(-1)^{\binom{n+1}{2}} q^{3\binom{n+1}{3}} \prod_{j=1}^{n} \left(1 - q^j\right)^{n+1-j}.$$

2. *For every* $n \geq 0$, $H_n(\tilde{\alpha}_k(x))$ *is a polynomial in* $x$ *of degree* $n(n+1)/2$ *with leading coefficient*

$$\left[x^{\frac{n(n+1)}{2}}\right] H_n\left(\widetilde{\alpha}_k(x)\right) = \alpha_0\alpha_1 \cdots \alpha_n(-1)^{\binom{n+1}{2}} q^{2\binom{n+1}{3}} \prod_{j=1}^{n} \left(1 - q^j\right)^{n+1-j}.$$

# The End