Hankel Determinants of Sequences Related to Bernoulli Polynomials, Euler Polynomials, and q-series

Lin Jiu

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Dalhouise Number Theory Seminar

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Acknowledgment

- S. Chern, L. Jiu, *S. Li*, and L. Wang, Leading coefficient in the Hankel determinants related to binomial and q-binomial transforms, Adv. Appl. Math. **176** (2026), Article 103051.
- L. Jiu and *Y. Li*, Hankel determinants of certain sequences of Bernoulli polynomials: A direct proof of an inverse matrix entry from Statistics, Contrib. Discrete Math. **19** (2024), 64–84.
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- K. Dilcher and L. Jiu, Hankel determinants of shifted sequences of Bernoulli and Euler numbers, *Contrib. Discrete Math.* **18** (2023), 146–175.
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Karl Dilcher

Ye Li

Shane Chern

Liuquan Wang

Shuhan Li

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Definition

The nth *Hankel determinants* of a given sequence $\mathrm{a} = (a_0, a_1, \ldots,)$ is the determinant of the nth *Hankel matrix*

$$H_n(\mathrm{a}) = H_n(a_k) = \det \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & a_3 & \cdots & a_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & a_{n+2} & \cdots & a_{2n} \end{pmatrix}$$

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Theorem

$H_n(\mathrm{C}) = 1$ *for all* $n = 0, 1, \ldots$

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The *Bernoulli polynomials* $B_n(x)$ and *Euler polynomials* $E_n(x)$ are given by

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!} \quad \text{and} \quad \frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}.$$

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Theorem (Al-Salam and Carlitz)

$$H_n(B_k)=(-1)^{\binom{n+1}{2}}\prod_{k=1}^{n}\frac{(k!)^6}{(2k)!(2k+1)!} \quad \textit{and} \quad H_n(E_k)=(-1)^{\binom{n+1}{2}}\prod_{k=1}^{n}(k!)^2.$$

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$$H_n(B_k(x))=(-1)^{\binom{n+1}{2}}\prod_{k=1}^{n}\frac{(k!)^6}{(2k)!(2k+1)!}=H_n(B_k).$$

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$$(B_n)_{n=0}^{\infty} = \left(1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, -\frac{1}{30}, 0, \frac{5}{66}, 0, -\frac{691}{2730}\right)$$

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$B_{2k}((1+x)/2)$

$H_1\left(B_{2k}\left(\frac{1+x}{2}\right)\right)$	$-\frac{1}{12}x^2 + \frac{1}{45}$
$H_2\left(B_{2k}\left(\frac{1+x}{2}\right)\right)$	$-\frac{1}{540}x^6 + \frac{97}{18900}x^4 - \frac{11}{4725}x^2 + \frac{16}{55125}$
$H_3\left(B_{2k}\left(\frac{1+x}{2}\right)\right)$	$\frac{1}{42000}x^{12} - \frac{121}{441000}x^{10} + \frac{153}{154000}x^8 - \frac{17441}{12262250}x^6 + \frac{8369}{11036025}x^4 - \frac{1632}{9634625}x^2 + \frac{256}{18883865}$

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$B_2\left(\frac{1+x}{2}\right)$	$\frac{x^2}{4}-\frac{1}{2}$
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Theorem (S. Chern, L. J., S. Li, and L. Wang)

For $n \geq 0$, $H_n\left(B_{2k}\left(\frac{1+x}{2}\right)\right)$ is a polynomial in x of degree $n(n+1)$ with leading coefficient

$$\left[x^{n(n+1)}\right] H_n\left(B_{2k}\left(\frac{1+x}{2}\right)\right) = (-1)^{\binom{n+1}{2}} \prod_{j=1}^{n} \frac{(j!)^6}{(2j)!(2j+1)!}.$$

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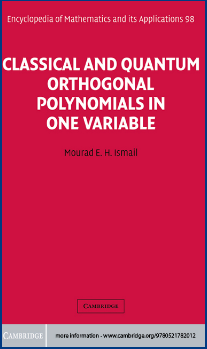

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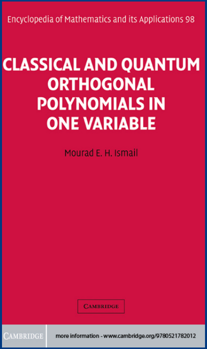

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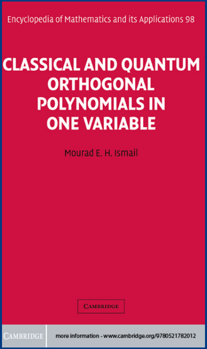

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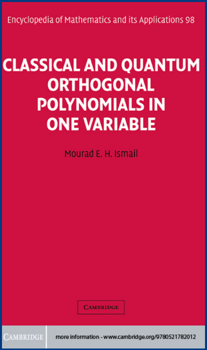

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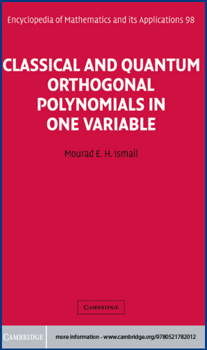

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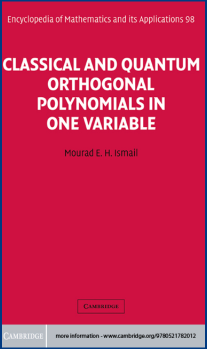

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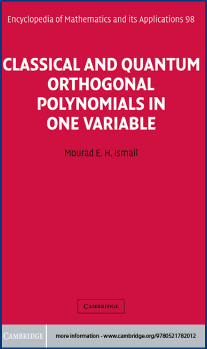

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$$P_{n+1} = (y+s_n)P_n(y) - t_nP_{n-1}(y) \Rightarrow \begin{cases} \sum_{n=0}^{\infty} a_nz^n = \cfrac{a_0}{1+s_0z-\cfrac{t_1z^2}{1+s_1z-\cfrac{t_2z^2}{\ddots}}} \\ H_n(\mathrm{a}) = a_0^{n+1}t_1^nt_2^{n-1}\cdots t_n \end{cases}$$

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&= \quad \frac{2}{1+\sqrt{1-4x}} = \frac{2}{2+\sqrt{1-4x}-1} \\
&= \quad \frac{1}{1-\frac{x}{c(x)}}
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Catalan Numbers $C_k = \binom{2k}{k}/(k+1)$

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Proposition (Canonical Contractions)

$$\frac{1}{1-\frac{\alpha_1 t}{1-\frac{\alpha_2 t}{1-\ddots}}} \overset{even}{=} \frac{1}{1-\alpha_1 t - \frac{\alpha_1\alpha_2 t^2}{1-(\alpha_2+\alpha_3)t-\frac{\alpha_3\alpha_4 t^2}{1-(\alpha_4+\alpha_5)t-\frac{\alpha_5\alpha_6 t^2}{\ddots}}}}$$

$$\overset{odd}{=} 1 + \frac{\alpha_1 t}{1-(\alpha_1+\alpha_2)t-\frac{\alpha_2\alpha_3 t^2}{1-(\alpha_3+\alpha_4)t-\frac{\alpha_4\alpha_5 t^2}{1-(\alpha_5+\alpha_6)t-\frac{\alpha_6\alpha_7 t^2}{\ddots}}}}.$$

Generalized Motzkin Numbers

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Given seqeunces σ_k and τ_k, the genarlized Motzkin numbers $M_{n,k}$ are defined by $M_{0,0} = 1$ and for $n > 0$

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Remark

If $\sigma_k = \tau_k = 1$, then $M_{n,0}$ are the Motzkin numbers.

Motzkin Paths

$M_{n,k}$ counts the number of $(1, \sigma_k, \tau_k)$-weighted Motzkin paths from $(0,0)$ to (n,k).

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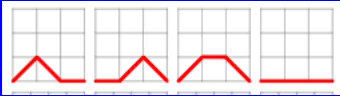

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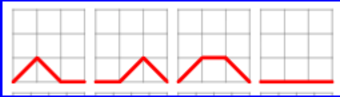

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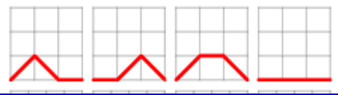

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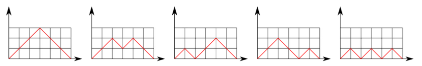

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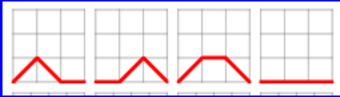

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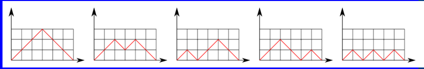

$$M_{5,0} = (-1)^3 \left(3^2 2^2 1^2 + 2^2 2^2 1^2 + 1^2 2^2 1^2 + 2^2 1^2 1^2 + 1^2 1^2 1^2\right)$$

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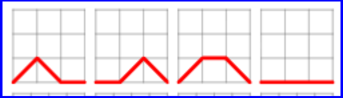

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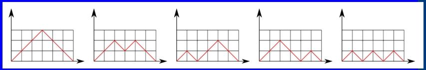

$$\begin{aligned} M_{5,0} &= (-1)^3 \left(3^2 2^2 1^2 + 2^2 2^2 1^2 \right. \\ &\quad \left. + 1^2 2^2 1^2 + 2^2 1^2 1^2 + 1^2 1^2 1^2\right) \\ &= -61 \end{aligned}$$

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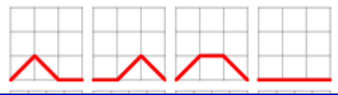

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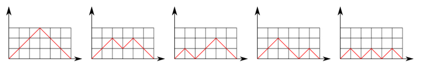

$$\begin{aligned} M_{5,0} &= (-1)^3 \left(3^2 2^2 1^2 + 2^2 2^2 1^2 \right. \\ &\quad \left. + 1^2 2^2 1^2 + 2^2 1^2 1^2 + 1^2 1^2 1^2\right) \\ &= -61 = E_6. \quad \text{(6th Euler number)} \end{aligned}$$

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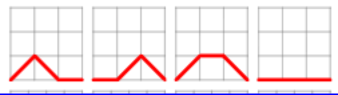

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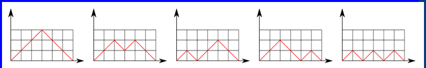

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Remark

Those 5 Dyck paths are counted by the Catalan number $C_3 = 5$.

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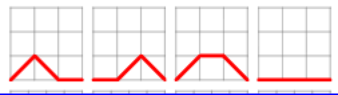

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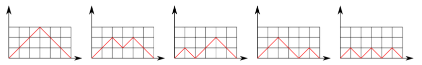

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Remark

Those 5 Dyck paths are counted by the Catalan number $C_3 = 5$.The general connection between C_n and E_{2n} exists.

How to compute/prove it?

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1. Generating function

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1. Generating function —Continued Fractions

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Theorem (K. Dilcher and L. J.)

$$\sum_{k=0}^{\infty} B_{2k+1}\left(\frac{x+1}{2}\right) z^{2k} = \frac{1}{2z^2}\left(\psi'\left(\frac{1}{z}+\frac{1-x}{2}\right) - \psi'\left(\frac{1}{z}+\frac{1+x}{2}\right)\right)$$

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$$= \cfrac{\frac{x}{2}}{1+\frac{(1-x^2)}{4}z^2 - \cfrac{\frac{(1^2-x^2)1^4}{4\cdot 1\cdot 3}z^4}{1+\frac{(5-x^2)}{4}z^2 - \cfrac{\frac{(2^2-x^2)2^4}{4\cdot 3\cdot 5}z^4}{1+\frac{(13-x^2)}{4}z^2 - \ddots}}}$$

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Hence,

$$H_n\left(B_{2k+1}\left(\frac{x+1}{2}\right)\right) = (-1)^{\binom{n+1}{2}}\left(\frac{x}{2}\right)^{n+1}\prod_{\ell=1}^{n}\left(\frac{\ell^4\left(x^2-\ell^2\right)}{4(2\ell+1)(2\ell-1)}\right)^{n+1-\ell}.$$

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Remark

We also have the orthogonal polynomials w.r.t. $E_{2k+\nu}\left(\frac{x+1}{2}\right)$, $\nu = 0, 1, 2$.

Linear Operator?

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GOAL:

$$\sum_{n=0}^{\infty} a_n z^n \stackrel{?}{=} \cfrac{a_0}{1 + s_0 z - \cfrac{t_1 z^2}{1 + s_1 z - \cfrac{t_2 z^2}{\ddots}}}$$

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- Identify the corresponding orthogonal polynomial $P_0 = 1$, $P_1 = y + s_0$, and

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- (HARD STEP) Define a proper linear operator L such that

$$L(y^n) = a_n \quad \text{and} \quad L(P_n) = 0 \quad \text{for} \quad n \geq 1.$$

q-Euler

Definition

The q-Euler numbers were introduced by Carlitz as

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Theorem (S. Chern and L. J.)

$$\det_{0 \le i,j \le n} \left(\epsilon_{i+j}\right) = \frac{(-1)^{\binom{n+1}{2}} q^{\frac{1}{4}\binom{2n+2}{3}}}{(1-q)^{n(n+1)}} \prod_{k=1}^{n} \frac{\left(q^2, q^2; q^2\right)_k}{\left(-q, -q^2, -q^2, -q^3; q^2\right)_k}$$

$$\det_{0 \le i,j \le n} \left(\epsilon_{i+j+1}\right) = \frac{(-1)^{\binom{n+2}{2}} q^{\frac{1}{4}\binom{2n+4}{3}}}{(1-q)^{n(n+1)} \left(1+q^2\right)^{n+1}} \prod_{k=1}^{n} \frac{\left(q^2, q^4; q^2\right)_k}{\left(-q^2, -q^3, -q^3, -q^4; q^2\right)_k}$$

$$\det_{0 \le i,j \le n} \left(\epsilon_{i+j+2}\right) = \frac{(-1)_2^{(n+2)} q^{\frac{1}{4}\binom{2n+4}{3}} (1+q)^n \left(1-(-1)^n q^{(n+2)^2}\right)}{(1-q)^{n(n+1)} \left(1+q^2\right)^{2(n+1)} \left(1+q^3\right)^{n+1}} \times \prod_{k=1}^{n} \frac{\left(q^4, q^4; q^2\right)_k}{\left(-q^3, -q^4, -q^4, -q^5; q^2\right)_k}$$

Remarks

- $$(A;q)_n := \prod_{k=1}^{n}(1-Aq^{k-1}), \quad (A_1,A_2,\ldots,A_n,q) := \prod_{j=1}^{n}(A_j;q)_n$$

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- $$\lim_{q\to 1}\det_{0\le i,j\le n}(\epsilon_{i+j}) = \left(-\frac{1}{16}\right)^{\binom{n+1}{2}}\prod_{k=1}^{n}((2k)!!)^2 = \left(-\frac{1}{4}\right)^{\binom{n+1}{2}}\prod_{k=1}^{n}(k!)^2.$$

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- We also use the big q-Jacobi polynomials

$$\mathscr{J}_{\ell,n}(z) := {}_3\phi_2\left(\begin{matrix} q^{-n}, -q^{n+\ell+1}, z \\ q^{\ell+1}, 0 \end{matrix}; q, q\right)$$

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- The linear operator is defined by

$$\Phi\left(\frac{1}{[n]_q!}\prod_{k=1}^{n}\left([k]_q + q^k z\right)\right) = \frac{1}{(-q^2;q)_n}, \qquad \text{where } [n]_q = \frac{1-q^n}{1-q}.$$

2. From Known Sequence(s)

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$$H_n(B_k(x)) = (-1)^{\binom{n+1}{2}} \prod_{k=1}^{n} \frac{(k!)^6}{(2k)!(2k+1)!} = H_n(B_k).$$

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Proposition

$$c_k(x) := \sum_{\ell=0}^{k} \binom{k}{\ell} c_\ell x^{k-\ell} \Rightarrow H_n(c_k) = H_n(c_k(x)).$$

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Proposition (L. J. and D. Shi)

Let z be a constant.

	Ran. Vari.	*Moments*	*Orthogonal Polynomials*
IF	X	$\mathbb{E}[X^n] = c_n$	$P_{n+1} = (y + s_n)P_n(y) - t_n P_{n-1}(y)$
THEN	$X + z$	$\sum_{\ell=0}^{k} \binom{k}{\ell} c_\ell z^{k-\ell}$	$\bar{P}_{n+1}(y) = (y + s_n - z)\,\bar{P}_n(y) - t_n \bar{P}_{n-1}(y)$

q-Binomial

Problem

What if

$$\binom{k}{\ell} \mapsto \begin{bmatrix} k \\ \ell \end{bmatrix}_q := \frac{(1-q^k)(1-q^{k-1})\cdots(1-q^{k-\ell+1})}{(1-q)(1-q^2)\cdots(1-q^\ell)}?$$

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Theorem (The q-binomial theorem)

$$\prod_{k=0}^{n-1}(1+q^k t) = \sum_{k=0}^{n} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q t^k.$$

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Remark

The binomial transform is symmetric between c_ℓ and $x^{k-\ell}$.

$$c_k(x) := \sum_{\ell=0}^{k} \binom{k}{\ell} c_\ell x^{k-\ell} = \sum_{\ell=0}^{k} \binom{k}{\ell} c_{k-\ell} x^\ell$$

Two problems

Problem

What are $H_n(\alpha_k(x))$ and $H_n(\tilde{\alpha}_k(x))$, where

$$\alpha_k(x) := \sum_{\ell=0}^{k} q^{\binom{\ell}{2}} \begin{bmatrix} k \\ \ell \end{bmatrix}_q \alpha_{k-\ell} x^\ell \quad \text{and} \quad \widetilde{\alpha}_k(x) := \sum_{\ell=0}^{k} q^{\binom{\ell}{2}} \begin{bmatrix} k \\ \ell \end{bmatrix}_q \alpha_\ell x^{k-\ell}?$$

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Remark

For $\tilde{\alpha}_k(x)$, we can define $\beta_\ell = q^{\binom{\ell}{2}} \alpha_\ell$.

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For $\tilde{\alpha}_k(x)$, we can define $\beta_\ell = q^{\binom{\ell}{2}} \alpha_\ell$.

Theorem (S. Chern, L. J., S. Li, and L. Wang)

1. *For every $n \geq 0$, $H_n(\alpha_k(x))$ is a polynomial in x of degree $n(n+1)$ with leading coefficient*

$$\left[x^{n(n+1)}\right] H_n\left(\alpha_k(x)\right) = \alpha_0^{n+1} (-1)^{\binom{n+1}{2}} q^{3\binom{n+1}{3}} \prod_{j=1}^{n} \left(1 - q^j\right)^{n+1-j}.$$

2. *For every $n \geq 0$, $H_n(\tilde{\alpha}_k(x))$ is a polynomial in x of degree $n(n+1)/2$ with leading coefficient*

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2.2 Shifts

Proposition (L. Mu, Y. Wang, Y. Yeh (2017))

Let P_n be the monic orthogonal polynomial w.r.t. sequence c_k, with recurrence $P_{n+1} = (y + s_n)P_n - t_n P_{n-1}$. Define

$$J := \begin{pmatrix} -s_0 & 1 & 0 & 0 & \cdots \\ t_1 & -s_1 & 1 & 0 & \cdots \\ 0 & t_2 & -s_2 & 1 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

and let J_n be the $(n+1)$th leading principal submatrix if J, with $d_n := \det J_n$. Then,

$$H_n(c_{k+1}) = H_n(c_k)d_n,$$

$$H_n(c_{k+2}) = H_n(c_k) \cdot \left(\prod_{\ell=1}^{n+1} t_\ell\right) \cdot \sum_{\ell=-1}^{n} \frac{d_\ell^2}{\prod_{j=1}^{\ell+1} t_j}.$$

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Remark

Shift to the left: $(c_0, c_1, c_2, \ldots) \to (c_1, c_2, \ldots,) \to (c_2, c_3, \ldots)$

Shift to the right

Lemma (K. Dilcher and L. J.)

Let P_n *be the monic orthogonal polynomial w.r.t. sequence* c_k*, with recurrence* $P_{n+1} = (y + s_n)P_n - t_nP_{n-1}$*. Define, for some constant* α*,*

$$b_k = \begin{cases} \alpha, & k = 0; \\ c_{k-1}, & k \geq 1. \end{cases} \quad ((c_0, c_1, \ldots) \to (\alpha, c_0, c_1, \ldots))$$

Then for $n \geq 2$*,*

$$\frac{H_{n+1}(b_k)}{H_n(c_k)} = -s_n \frac{H_n(b_k)}{H_{n-1}(c_k)} - t_n \frac{H_{n-1}(b_k)}{H_{n-2}(c_k)}.$$

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Proposition (K. Dilcher and L. J.)

Define

$$b_k(x) := \begin{cases} 0, & k=0; \\ E_{2k-2}, & k \geq 1, \end{cases}$$

then for $n \geq 1$,

$$H_n(b_k(x)) = \frac{(-1)^n}{4^n (n!)^2} \left(\sum_{j=0}^{n-1} \frac{16^j}{(2j+1)^2 \binom{2j}{j}^2} \right) H_n(E_{2k}).$$

Remarks

1. In general, the following Hankel determinants are "independent"

$$H_n(c_k), H_n(c_{2k}), H_n(c_{2k+1}), H_n(kc_k), H_n((k+1)c_k), H_n\left(\frac{c_{k+1}}{k+1}\right), \ldots$$

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Theorem (L. J., Y. Li, and C. Krattenthaler)

$$H_n\left(\frac{B_{2k+5}\left(\frac{x+1}{2}\right)}{2k+5}\right)$$
$$=\frac{\prod_{i=1}^{n}\frac{(2i+3)!^2(2i+2)!^2}{(4i+5)!(4i+4)!}\prod_{\ell=0}^{n}(x-2n-1+2\ell)_{4n-4\ell+3}}{5\cdot 2^{n+2}}$$
$$\times\sum_{i=1}^{n+2}\frac{(2i-1)\left(n+\frac{5}{2}\right)_{i-1}\left(\frac{x+1}{2}\right)_{n+2}\left(\frac{x-2n-3}{2}\right)_{n+2}}{\left(n-i+\frac{5}{2}\right)_i(n+2-i)!(n+1+i)!\left(x^2-(2i-1)^2\right)}.$$

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2. It is almost impossible to find the recurrences of $H_n(c_k)$ (unless the sequence c_k is easy to handle).
3. There is an interesting paper: Alan Sokal, A simple algorithm for expanding a power series as a continued fraction, Expositiones Mathematica, **41** (2023), 245–287.

Remarks

4. It seems that the following Hankel matrix is of interests: define $I_k := \sum_{c=1}^{r} c^k$, and consider

$$V_n := \begin{pmatrix} I_0 & I_2 & \cdots & I_{2n} \\ I_2 & I_4 & \cdots & I_{2n+2} \\ \vdots & \vdots & \ddots & \vdots \\ I_{2n} & I_{2n+2} & \cdots & I_{4n} \end{pmatrix}.$$

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V_n is invertible iff $n < r$.

Dear colleague,

I am a researcher who mainly studies questions from numerical analysis. I have run into the following problem :
- consider the Hankel matrix whose k-th element is $B_k(x)/k$ (in my case x is a positive integer).
I would like to show that the above Hankel matrix is positive definite whatever the size.

I have gone through some of your work, in particular the one where you collect many formulas on Hankel determinants that are related to Bernoulli and Euler polynomials. I have not seen the above one but your work is so extensive and I have so little experience in that field that I thought it might be easier to ask you directly if the formula for this Hankel determinant is known.

I would be most grateful if you could help me by pointing towards a reference in case the result is alkready known.

What have we computed

$$\begin{array}{cccc}
B_{2n+1}\left(\frac{x+1}{2}\right) & E_{2k}\left(\frac{x+1}{2}\right) & E_{2k+1}\left(\frac{x+1}{2}\right) & E_{2k+2}\left(\frac{x+1}{2}\right) \\
B_k\left(\frac{x+r}{q}\right) \pm B_k\left(\frac{x+s}{q}\right) & B_{k,\chi_3} & B_{k,\chi_4} & B_{k,\chi_6} \\
E_k\left(\frac{x+r}{q}\right) \pm E_k\left(\frac{x+s}{q}\right) & (2k+2)E_{2k+1}(1) & kE_{k-1}(x) & (2k+1)E_{2k} \\
\frac{B_{k,\chi_{6,1}}}{k+1} & \frac{B_{k,\chi_{6,2}}}{k+1} & \frac{B_{k,\chi_{8,1}}}{k+1} & \frac{B_{k,\chi_{8,2}}}{k+1} \\
(2k+1)B_{2k}\left(\frac{1}{2}\right) & (2k+3)B_{2k} & \frac{B_{2k+1}\left(\frac{x+1}{2}\right)}{2k+1} & \frac{B_{2k+3}\left(\frac{x+1}{2}\right)}{2k+3} \\
\frac{B_{2k+5}\left(\frac{x+1}{2}\right)}{2k+5} & \epsilon_k & \epsilon_{k+1} & \epsilon_{k+2}
\end{array}$$

where

$$\epsilon_n := \frac{1}{(1-q)^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1+q}{1+q^{k+1}}$$

What have we computed

$B_{2n+1}\left(\frac{x+1}{2}\right)$	$E_{2k}\left(\frac{x+1}{2}\right)$	$E_{2k+1}\left(\frac{x+1}{2}\right)$	$E_{2k+2}\left(\frac{x+1}{2}\right)$
$B_k\left(\frac{x+r}{q}\right) \pm B_k\left(\frac{x+s}{q}\right)$	B_{k,χ_3}	B_{k,χ_4}	B_{k,χ_6}
$E_k\left(\frac{x+r}{q}\right) \pm E_k\left(\frac{x+s}{q}\right)$	$(2k+2)E_{2k+1}(1)$	$kE_{k-1}(x)$	$(2k+1)E_{2k}$
$\frac{B_{k,\chi_{6,1}}}{k+1}$	$\frac{B_{k,\chi_{6,2}}}{k+1}$	$\frac{B_{k,\chi_{8,1}}}{k+1}$	$\frac{B_{k,\chi_{8,2}}}{k+1}$
$(2k+1)B_{2k}(\frac{1}{2})$	$(2k+3)B_{2k}$	$\frac{B_{2k+1}\left(\frac{x+1}{2}\right)}{2k+1}$	$\frac{B_{2k+3}\left(\frac{x+1}{2}\right)}{2k+3}$
$\frac{B_{2k+5}\left(\frac{x+1}{2}\right)}{2k+5}$	ϵ_k	ϵ_{k+1}	ϵ_{k+2}

where

$$\epsilon_n := \frac{1}{(1-q)^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1+q}{1+q^{k+1}}$$

$$\tilde{\alpha}_k^{u,v}(x) := \sum_{\ell=0}^{k}\begin{bmatrix} k \\ \ell \end{bmatrix}_q (u;q)_\ell v^\ell x^{k-\ell} = \sum_{\ell=0}^{k}\begin{bmatrix} k \\ \ell \end{bmatrix}_q (u;q)_{k-\ell} v^{k-\ell} x^\ell.$$

What have we computed

$b_k, k \geq 1$	b_0	Prop.	$b_k, k \geq 1$	b_0	Prop.
B_{k-1}	0	3.1	$E_{k+3}(1)$	$(\frac{-1}{4})$	5.2
B_{2k}	(1)	6.1	$E_{2k-1}(1)$	0	3.3
$(2k+1)B_{2k}$	(1)	6.2	$E_{2k+5}(1)$	$(\frac{1}{2})$	5.1
$(2^{2k}-1)B_{2k}$	(0)	3.4	$E_k(1)/k!$	(1)	3.6
$(2k+1)E_{2k}$	0	3.5	$E_{2k-1}(1)/(2k-1)!$	0	6.3
E_{2k-2}	0	7.3	$E_{2k-2}(\frac{x+1}{2})$	0	7.2
$E_{k-1}(1)$	0	3.2	$E_{k-1}^{(p)}$	$\alpha \in \mathbb{R}$	8.1

TABLE 2. Summary of results.

What have we computed

$b_k, k \geq 1$	b_0	Prop.	$b_k, k \geq 1$	b_0	Prop.
B_{k-1}	0	3.1	$E_{k+3}(1)$	$(\frac{-1}{4})$	5.2
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TABLE 2. Summary of results.

Degrees and the leading coefficients of $H_n(c_k)$ for c_k being

$$B_{2k}\left(\frac{x+1}{2}\right), \alpha_k(x) := \sum_{\ell=0}^{k} q^{\binom{\ell}{2}} \begin{bmatrix} k \\ \ell \end{bmatrix}_q \alpha_{k-\ell} x^\ell$$

$$\widetilde{\alpha}_k(x) := \sum_{\ell=0}^{k} q^{\binom{\ell}{2}} \begin{bmatrix} k \\ \ell \end{bmatrix}_q \alpha_\ell x^{k-\ell}$$

Next?

Next?

1. Other sequences.

Next?

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1. Other sequences. (e.g. q-series) Or, unsolved ones

$H_1\left(B_{2k}\left(\frac{1+x}{2}\right)\right)$	$-\frac{1}{12}x^2+\frac{1}{45}$
$H_2\left(B_{2k}\left(\frac{1+x}{2}\right)\right)$	$-\frac{1}{540}x^6+\frac{97}{18900}x^4-\frac{11}{4725}x^2+\frac{16}{55125}$
$H_3\left(B_{2k}\left(\frac{1+x}{2}\right)\right)$	$\frac{1}{42000}x^{12}-\frac{121}{441000}x^{10}+\frac{153}{154000}x^8-\frac{17441}{12262250}x^6+\frac{8369}{11036025}x^4-\frac{1632}{9634625}x^2+\frac{256}{18883865}$

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Coefficients in $H_2\left(\widetilde{\alpha}_k(x)\right)$:

x^3	$-q^2(1-q)^3(1+q)\alpha_0\alpha_1\alpha_2$
x^2	$q^2(1-q)^2(1+q)\left[\left(q+q^2\right)\alpha_0\alpha_1\alpha_3-\alpha_0\alpha_2^2-\alpha_1^2\alpha_2\right]$
x^1	$-q^2(1-q)\,[\text{something}]$
x^0	$q^3\left[q^4\alpha_0\alpha_2\alpha_4-q^3\alpha_0\alpha_3^2+2q\alpha_1\alpha_2\alpha_3-q^3\alpha_1^2\alpha_4-\alpha_2^3\right]$

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2. More general results

Next?

- Continued fractions

Next?

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- Generalized Motzkin numbers and lattice paths

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- Orthogonal polynomials

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Table 2
$b_n^{(p)}$ for $1 \le n,\ p \le 5$.

	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$
$n=1$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{5}{12}$
$n=2$	$\frac{4}{15}$	$\frac{13}{30}$	$\frac{3}{5}$	$\frac{23}{30}$	$\frac{14}{15}$
$n=3$	$\frac{81}{140}$	$\frac{372}{455}$	$\frac{1339}{1260}$	$\frac{2109}{1610}$	$\frac{1527}{980}$
$n=4$	$\frac{64}{63}$	$\frac{3736}{2821}$	$\frac{138688}{84357}$	$\frac{668543}{339549}$	$\frac{171830}{74823}$
$n=5$	$\frac{625}{396}$	$\frac{1245075}{636988}$	$\frac{299594775}{127670972}$	$\frac{42601023200}{15509529057}$	$\frac{3638564965}{1154491404}$

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Conjecture (K. Dilcher (2018))

$$b_1^{(p)} = \frac{p}{12},$$

$$b_2^{(p)} = \frac{5p+3}{10},$$

$$b_3^{(p)} = \frac{175p^2+315p+158}{140(2p+3)},$$

$$b_4^{(p)} = \frac{6125p^4+25725p^3+41965p^2+29547p+7230}{21(5p+3)(175p^2+315p+158)}, \qquad \textit{as well as } b_5^{(p)}.$$

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Proposition (K. Dilcher and L. J.(Feb. 10th, 2026))

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Bernoulli polynomials of higher-orders

Definition

The Bernoulli polynomials of order p, denoted by $B_n^{(p)}(x)$, are defined by

$$\left(\frac{z}{e^z - 1}\right)^p e^{zx} = \sum_{n=0}^{\infty} B_n^{(p)}(x) \frac{z^n}{n!}.$$

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We let $\varrho_n^{(p)}$ be the monic orthogonal polynomial w.r.t. $B_n^{(p)}(x)$, and assume the three-term recurrence is

$$\varrho_{n+1}^{(p)}(y) = \left(y - a_n^{(p)}\right)\varrho_n^{(p)}(y) + b_n^{(p)}\varrho_{n-1}^{(p)}(y).$$

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Remark

For $p = 1$, it is the case of Bernoulli polynomials $B_n(x)$ and the corresponding orthogonal polynomial are proportional to the continuous Hahn polynomial $p_n(y; a, b, c, d)$, as

$$\varrho_n^{(1)}(y) = \frac{n!}{(n+1)_n} p_n\left(y; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right).$$

The End

Thank you!