Identities Involving Central (q-)Binomial Coefficients via (q-)Hypergeometric Series

Lin Jiu

Zu Chongzhi Center, Duke Kunshan University
Dalhouise Number Theory Seminar

Mar. 18th, 2026

Joint work with Shane Chern (middle) and Karl Dilcher (right)

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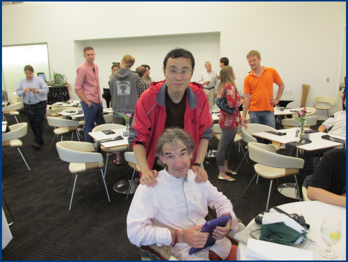
- To summarize the ideas and practice on my SIAM meeting in July.

Recent Results by Karl and Christophe Vignat



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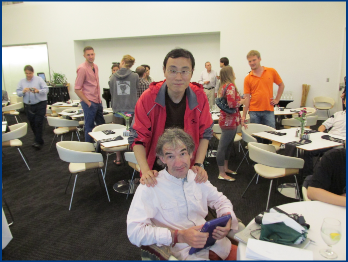

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ArXiv:2511.00109

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We begin here with the pair of series

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+1)} = \frac{\pi}{2}, \qquad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+2)} = 1. \tag{1.1}$$

Both identities can be obtained from the power series

$$\sum_{k=0}^{\infty} \binom{2k}{k} x^k = \frac{1}{\sqrt{1-4x}}, \tag{1.2}$$

which is a special case of the binomial theorem. For the first identity in (1.1) we replace x by $x^2/4$ in (1.2) and integrate; this gives a well-known series for the arcsine function which we evaluate at $x = 1$. Similarly, the second identity in (1.1) follows from integrating (1.2) as is. For details, see [6, p. 449–451].

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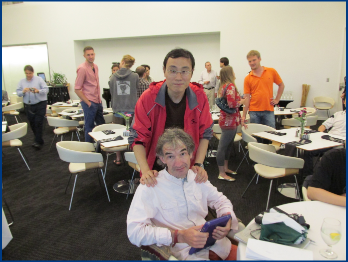

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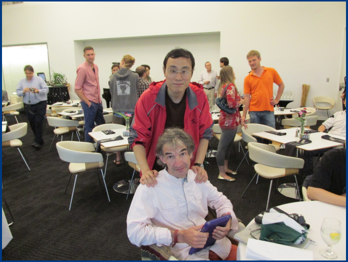

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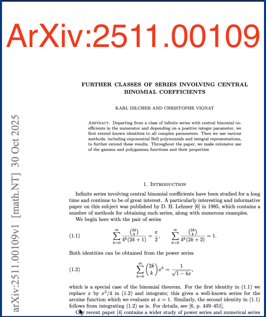

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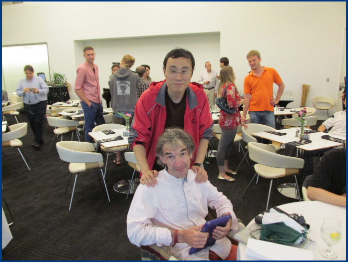

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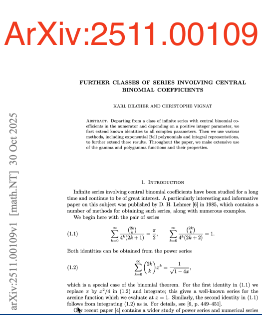

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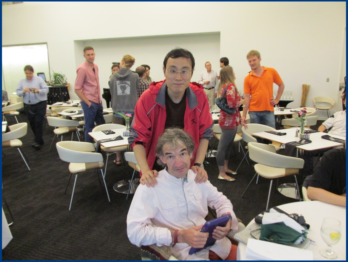

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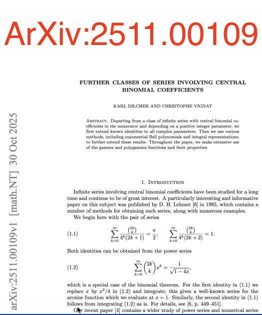

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For any $z\in\mathbb{C}\backslash\{-1,-3,-5,\ldots\}$,

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$$LHS = \frac{1}{1+z} \, {}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{z+1}{2} \\ \frac{z+3}{2} \end{matrix} ; 1 \right)$$

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$$\sum_{k=0}^{\infty} \frac{1}{4^k} \left[\begin{matrix} d-\frac{1}{2}, & d-\frac{1}{2}, & d-\frac{1}{2} \\ 1, & 1, & d \end{matrix} \right]_k (6k+2d-1) = \frac{2^{4d-2}}{\binom{2d-2}{d-1}\pi}.$$

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Here

$$\left[\begin{matrix} a & b & c \\ d & e & f \end{matrix} \right]_k := \frac{(a)_k (b)_k (c)_k}{(d)_k (e)_k (f)_k},$$

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Fact

$$LHS = {}_4F_3 \left(\begin{matrix} d-\frac{1}{2}, d-\frac{1}{2}, d-\frac{1}{2}, \frac{2d+5}{6} \\ 1, d, \frac{2d-1}{6} \end{matrix} ; \frac{1}{4} \right) (2d-1).$$

Hypergeometric Functions and Basic Hypergeometric Series

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Definition ((Generalized) Hypergeometric Series)

$$ {}_pF_q\left(\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}; z\right) := \sum_{k=0}^{\infty} \left[\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}\right]_k \frac{z^k}{k!}. $$

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Definition (q-Pochhammer Symbol)

$$ (a;q)_k = \begin{cases} 1, & k = 0; \\ \prod_{j=1}^{k} \left(1 - aq^{j-1}\right), & k \in \mathbb{N}; \\ \prod_{j=1}^{\infty} \left(1 - aq^{j-1}\right), & k = \infty. \end{cases} $$

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$(a_1;q)_k(a_2;q)_k \cdots (a_n;q)_k = (a_1, a_2, \ldots, a_n; q)_k$

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Definition (Basic Hypergeometric Function)

$${}_r\phi_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; q, z\right) = \sum_{n=0}^{\infty} \left((-1)^n q^{\binom{n}{2}}\right)^{1+s-r} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s, q; q)_n} z^n.$$

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In particular, if $r = s + 1$,

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Theorem (The q-Gauss Summation)

For $|c/(ab)| < 1$,

$$ {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, \frac{c}{ab}\right) = \frac{\left(\frac{c}{a}; q\right)_\infty \left(\frac{c}{b}; q\right)_\infty}{(c; q)_\infty \left(\frac{c}{ab}; q\right)_\infty}. $$

The q-analog of the 1st Dilcher-Vignat Identity

Theorem

For $z \in \mathbb{C} \setminus \{-1 \pm 2\pi ik/\log q, -3 \pm 2\pi ik/\log q, \ldots\}$, *where* k *is a nonnegative integer,*

$$\sum_{k=0}^{\infty} \frac{\begin{bmatrix} 2k \\ k \end{bmatrix}_{q^2}}{\left[\frac{2k+1+z}{2}\right]_{q^2}} \cdot \frac{q^k}{(-q;q)_{2k}} = \frac{\Gamma_{q^2}\left(\frac{z+1}{2}\right)}{\Gamma_{q^2}\left(\frac{z+2}{2}\right)} \Gamma_{q^2}\left(\frac{1}{2}\right).$$

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1. The q-integer $[n]_q := 1 + q + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$.

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For $z \in \mathbb{C} \backslash \{-1 \pm 2\pi ik/\log q, -3 \pm 2\pi ik/\log q, \ldots\}$, where k is a nonnegative integer,

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$$\sum_{k=0}^{\infty} \frac{\begin{bmatrix}2k\\k\end{bmatrix}_{q^2}}{(-q;q)_{2k}} \frac{q^k}{1-aq^{2k}} = \frac{(q^2, aq; q^2)_\infty}{(q; a; q^2)_\infty}.$$

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(2) follows directly from the lemma.

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Theorem (K. Dilcher and C. Vignat)

For all $m \geq 0$

$$\sum_{\substack{k=0\\k\neq m}}^{\infty} \frac{\binom{2k}{k}}{4^k(2k-2m)} = \binom{2m}{m} \cdot \frac{\log 2 - H_{2m} + H_m}{4^m},$$

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For any $m \in \mathbb{N}$,

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Theorem (G. Andrews and M. Merca, The truncated pentagonal number theorem)

$$\frac{1}{(q;q)_\infty}\sum_{j=0}^{k-1}(-1)^j q^{j(3j+1)/2}\left(1-q^{2j+1}\right) = 1+(-1)^{k-1}\sum_{n=1}^{\infty}\frac{q^{\binom{k}{2}+(k+1)n}}{(q;q)_n}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q .$$

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Theorem (Euler's pentagonal number theorem)

$$\prod_{j\geq 1}(1-q^j) = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n+1)}{2}}$$

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Fact

By far, we failed to find the closed form of the finite sum above.

$$\sum_{\substack{k=0\\k\neq m}}^{\infty} \frac{\begin{bmatrix}2k\\k\end{bmatrix}_{q^2}}{[k-m]_{q^2}} \cdot \frac{q^k}{(-q;q)_{2k}} = \frac{\begin{bmatrix}2m\\m\end{bmatrix}_{q^2} q^m \left(1-q^2\right)}{(-q;q)_{2m}} \sum_{k=2m+1}^{\infty} \frac{(-1)^{k-1} q^k}{1-q^k}.$$

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Mathematica and OEIS suggest

$$\sum_{k=0}^{m-1} \frac{\binom{2k}{k}}{4^k(2k-2m)} = -\frac{\binom{2m}{m}}{4^m} \sum_{j=0}^{m-1} \frac{1}{2j+1}.$$

$$\frac{4^m}{\binom{2m}{m}} \sum_{k=0}^{m-1} \frac{\binom{2k}{k}}{4^k(k-m)} = -2 \sum_{j=0}^{m-1} \frac{1}{2j+1}$$

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Then I "gave it up" and used the WZ-method to prove both.

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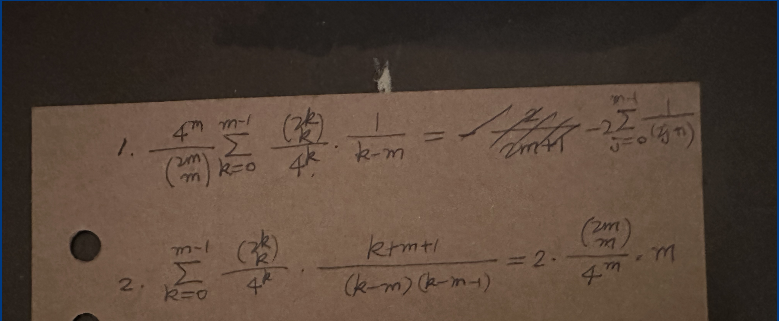

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Study a family of the finite sums: for $\nu \in \mathbb{N}$

$$S_\nu(m) := \sum_{k=0}^{m-1} \frac{\binom{2k}{k}}{4^k} \cdot \frac{(k+m+1)\cdots(k+m+\nu-1)}{(k-m)(k-m-1)\cdots(k-m-\nu+1)}.$$

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Proposition

$$S_2(m) = \frac{\binom{2m}{m}}{4^m} \cdot 2m,$$

$$S_4(m) = \frac{\binom{2m}{m}}{4^m} \cdot \frac{2m}{3^2}\left(2m^2+3m+4\right),$$

$$S_6(m) = \frac{\binom{2m}{m}}{4^m} \cdot \frac{2m}{2\cdot 3^2\cdot 5^2}\left(12m^4+60m^3+125m^2+125m+128\right),$$

$$S_8(m) = \frac{\binom{2m}{m}}{4^m} \cdot \frac{2m}{2\cdot 3^2\cdot 5^2\cdot 7^2}\left(40m^6+420m^5+1834m^4+4263m^3 +5887m^2+4998m+4608\right).$$

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Remark

WZ-method can finish the proof.

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Fact

The recurrence obtained by WZ-method can finish the proof by induction

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Fact

The recurrence obtained by WZ-method can finish the proof by induction if one can guess a formula.

q-analog

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How about

$$\sum_{\substack{k=0 \\ k\neq m}}^{\infty} \frac{\begin{bmatrix} 2k \\ k \end{bmatrix}_{q^2}}{[k-m]_{q^2}} \cdot \frac{q^k}{(-q;q)_{2k}} = \frac{\begin{bmatrix} 2m \\ m \end{bmatrix}_{q^2} q^m \left(1-q^2\right)}{(-q;q)_{2m}} \sum_{k=2m+1}^{\infty} \frac{(-1)^{k-1} q^k}{1-q^k}?$$

In fact, the original form was

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Problem (Is the following identity true?)

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Proposition

For all $m \in \mathbb{N}$,

$$S_{1,q}(m) := \sum_{k=0}^{m-1} \frac{\begin{bmatrix}2k\\k\end{bmatrix}_{q^2}}{(-q;q)_{2k}} \frac{q^k}{[k-m]_{q^2}} = -\frac{\begin{bmatrix}2m\\m\end{bmatrix}_{q^2} q^m(1-q^2)}{(-q;q)_{2m}} \sum_{j=0}^{m-1} \frac{q^{2j+1}}{1-q^{2j+1}}.$$

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Shane gave the proof in the train from D.C. to New York, after the Joint Meeting.

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Definition

$$S_{\nu,q}(m) := \sum_{k=0}^{m-1} \frac{\begin{bmatrix} 2k \\ k \end{bmatrix}_{q^2}}{(-q;q)_{2k}} \cdot \frac{q^k [k+m+1]_{q^2} \cdots [k+m+\nu-1]_{q^2}}{[k-m]_{q^2} [k-m-1]_{q^2} \cdots [k-m-\nu+1]_{q^2}}.$$

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For all $m \in \mathbb{N}$,

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Problem

How?

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&= (2m+2) \left(S_1(m+1) - \frac{\binom{2m}{m}}{4^m} \cdot \frac{1}{m-(m+1)} \right) \\
&\quad -(2m+1) S_1(m).
\end{aligned}
$$

q-PFD

Lemma

For $\nu \geq 1$ *and* $k \neq m, m+1, \ldots, m+\nu-1$,

$$\sum_{j=0}^{\nu-1} \frac{(-1)^{\nu-1-j} q^{(\nu-1-j)(\nu-j)}}{1-q^{2k-2m-2j}} \begin{bmatrix} 2m+\nu-1+j \\ \nu-1 \end{bmatrix}_{q^2} \begin{bmatrix} \nu-1 \\ j \end{bmatrix}_{q^2} = \frac{(q^{2(k+m+1)}; q^2)_{\nu-1}}{(q^{2(k-m-\nu+1)}; q^2)_{\nu}}.$$

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Corollary

For $\nu \geq 1$ *and* $k \neq m, m+1, \ldots, m+\nu-1$,

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Theorem

$$S_{\nu,q}(m) = -\frac{\begin{bmatrix}2m\\ m\end{bmatrix}_{q^2} q^m}{(-q;q)_{2m}} \sum_{j=0}^{\nu-1} (-1)^{\nu-1-j} q^{(\nu-1-j)(\nu-j)} \begin{bmatrix}2m+\nu-1+j\\ \nu-1\end{bmatrix}_{q^2} \begin{bmatrix}\nu-1\\ j\end{bmatrix}_{q^2}$$

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By taking $q \to 1$, we have the following.

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Corollary

$$S_\nu(m) = \frac{\binom{2m}{m}}{4^m} \sum_{j=0}^{\nu-1} (-1)^{\nu-1-j} \binom{2m+\nu-1+j}{\nu-1} \binom{\nu-1}{j}$$
$$\left(\frac{(2m+1)(2m+3)\cdots(2m+2j-1)}{(2m+2)(2m+4)\cdots(2m+2j)} \sum_{s=0}^{m+j-1} \frac{2}{2s+1} \right.$$
$$\left. - \sum_{\ell=0}^{j-1} \frac{(2m+1)(2m+3)\cdots(2m+2\ell-1)}{(2m+2)(2m+4)\cdots(2m+2\ell)(\ell-j)} \right).$$

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$$\sum_{k=0}^{m-1} \frac{\binom{2k}{k}^2}{4^{2k}} \cdot \frac{1}{k+1} = 4m \cdot \frac{\binom{2m}{m}^2}{4^{2m}},$$

$$\sum_{k=0}^{m-1} \frac{\binom{2k}{k}^2}{4^{2k}} \cdot \frac{(2k+1)(4k+3)}{k+1} = 4m(2m+1) \cdot \frac{\binom{2m}{m}^2}{4^{2m}},$$

$$\sum_{k=0}^{m-1} \frac{\binom{2k}{k}^2}{4^{2k}} \cdot \frac{(2k+1)(2k+3)^2(4k+5)}{k+2}$$

$$= \frac{4m(2m+1)(2m+3)\left(2m^2+3m+4\right)}{3(m+1)} \cdot \frac{\binom{2m}{m}^2}{4^{2m}}.$$

q-analogues of the first two identities

$$\sum_{k=0}^{m-1} \frac{{2k \brack k}_{q^2}^2}{(-q;q)_{2k}^2} \frac{q^{2k}}{[2k+2]_q} = \frac{{2m \brack m}_{q^2}^2}{(-q;q)_{2m}^2} [2m]_q.$$

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- Karl claimed to me and Shane, that he has a family of infinitely many such identities.

q-analogues of the first two identities

$$\sum_{k=0}^{m-1} \frac{{2k \brack k}_{q^2}^2}{(-q;q)_{2k}^2} \frac{q^{2k}}{[2k+2]_q} = \frac{{2m \brack m}_{q^2}^2}{(-q;q)_{2m}^2} [2m]_q.$$

$$\sum_{k=0}^{m-1} \frac{{2k \brack k}_{q^2}^2}{(-q;q)_{2k}^2} \frac{q^{-k-1}[2k+1]_q[4k+3]_q}{[2k+2]_q} = \frac{{2m \brack m}_{q^2}^2}{(-q;q)_{2m}^2} q^{-m}[2m]_q[2m+1]_q.$$

- The q-analog of the third one is still missing.

  $$2m^2+3m+4 \mapsto [2m^2+3m+4]_q \text{ or } [2m^2+3m+4]_{q^2}$$

- Karl claimed to me and Shane, that he has a family of infinitely many such identities.
- $$\sum_{k=0}^{n} \frac{{2k \brack k}_{q^2}}{{\frac{2k+1+z}{2} \brack}_{q^2}} \cdot \frac{q^k}{(-q;q)_{2k}} = ?$$

The End

Thank you