Identities Involving Central (q-)Binomial Coefficients via (q-)Hypergeometric Series

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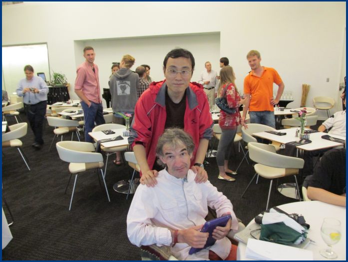
SIAM AN 26 — From Orthogonal Polynomials to Nonlinear Special Functions with Applications Part III

July 10th, 2026

Joint work with Shane Chern (middle) and Karl Dilcher (right)

Recent Results by Karl and Christophe Vignat



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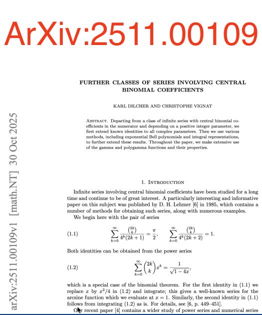

ArXiv:2511.00109

arXiv:2511.00109v1 [math.NT] 30 Oct 2025

FURTHER CLASSES OF SERIES INVOLVING CENTRAL BINOMIAL COEFFICIENTS

KARL DILCHER AND CHRISTOPHE VIGNAT

Abstract. Departing from a class of infinite series with central binomial coefficients in the numerator and depending on a positive integer parameter, we first extend known identities to all complex parameters. Then we use various methods, including exponential Bell polynomials and integral representations, to further extend these results. Throughout the paper, we make extensive use of the gamma and polygamma functions and their properties.

1. Introduction

Infinite series involving central binomial coefficients have been studied for a long time and continue to be of great interest. A particularly interesting and informative paper on this subject was published by D. H. Lehmer [6] in 1985, which contains a number of methods for obtaining such series, along with numerous examples.

We begin here with the pair of series

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{4^k(2k+1)}=\frac{\pi}{2},\qquad \sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{4^k(2k+2)}=1. \tag{1.1}$$

Both identities can be obtained from the power series

$$\sum_{k=0}^{\infty}\binom{2k}{k}x^k=\frac{1}{\sqrt{1-4x}}, \tag{1.2}$$

which is a special case of the binomial theorem. For the first identity in (1.1) we replace x by $x^2/4$ in (1.2) and integrate; this gives a well-known series for the arcsine function which we evaluate at $x=1$. Similarly, the second identity in (1.1) follows from integrating (1.2) as is. For details, see [6, p. 449–451].

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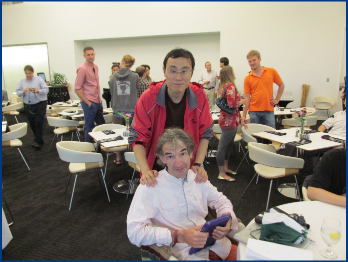

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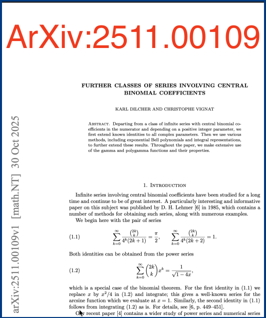

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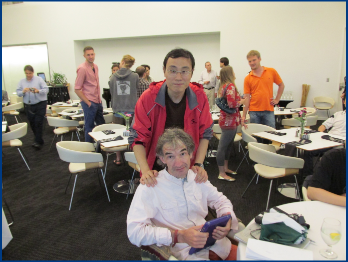

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Theorem (K. Dilcher and C. Vignat)

For any $z\in\mathbb{C}\backslash\{-1,-3,-5,\dots\}$,

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Thanks to the weekly discussion with John Campbell and Karl, I could immediately recognize

$$LHS = \frac{1}{1+z} \, {}_2F_1 \left(\begin{matrix} \frac{1}{2}, \frac{z+1}{2} \\ \frac{z+3}{2} \end{matrix} ; 1 \right)$$

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John Campbell

Hypergeometric Functions and Basic Hypergeometric Series

Definition ((Generalized) Hypergeometric Series)

$$
{}_pF_q\left(\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}; z\right) := \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \cdot \frac{z^k}{k!},
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where the Pochhammer symbol is defined by
$(a)_k := a(a+1)\cdots(a+k-1)$.

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Definition (q-Pochhammer Symbol)

$$ (a;q)_k = \begin{cases} 1, & k = 0; \\ \prod_{j=1}^{k}\left(1 - aq^{j-1}\right), & k \in \mathbb{N}; \\ \prod_{j=1}^{\infty}\left(1 - aq^{j-1}\right), & k = \infty. \end{cases} $$

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Remark

$(a_1;q)_k(a_2;q)_k \cdots (a_n;q)_k = (a_1, a_2, \ldots, a_n; q)_k$

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$$ {}_r\phi_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; q, z\right) = \sum_{n=0}^{\infty} \left((-1)^n q^{\binom{n}{2}}\right)^{1+s-r} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s, q; q)_n} z^n. $$

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Theorem (The q-Gauss Summation)

For $|c/(ab)| < 1$,

$$ {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, \frac{c}{ab}\right) = \frac{\left(\frac{c}{a}; q\right)_\infty \left(\frac{c}{b}; q\right)_\infty}{(c; q)_\infty \left(\frac{c}{ab}; q\right)_\infty}. $$

The q-analog of the 1st Dilcher-Vignat Identity

Theorem (S. Chern, K. Dilcher, and L.J.)

For $z \in \mathbb{C} \backslash \{-1 \pm 2\pi i k / \log q, -3 \pm 2\pi i k / \log q, \ldots\}$, *where* k *is a nonnegative integer,*

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4. The q-gamma function

$$\Gamma_q(x) := \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}$$

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+1+z)} = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(\frac{z+1}{2}\right)}{\Gamma\left(\frac{z+2}{2}\right)} \tag{1}$$

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- When $q \to 1$, (2) becomes (1).
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(2) follows directly from the lemma.

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For all $m \geq 0$

$$\sum_{\substack{k=0\\k\neq m}}^{\infty} \frac{\binom{2k}{k}}{4^k(2k-2m)} = \binom{2m}{m} \cdot \frac{\log 2 - H_{2m} + H_m}{4^m},$$

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Theorem (G. Andrews and M. Merca, The truncated pentagonal number theorem)

$$\frac{1}{(q;q)_\infty}\sum_{j=0}^{k-1}(-1)^j q^{j(3j+1)/2}\left(1-q^{2j+1}\right) = 1+(-1)^{k-1}\sum_{n=1}^{\infty}\frac{q^{\binom{k}{2}+(k+1)n}}{(q;q)_n}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q .$$

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Theorem (Euler's pentagonal number theorem)

$$\prod_{j\geq 1}(1-q^j) = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n+1)}{2}}$$

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we should ask whether there is a closed form of

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Fact

By far, we failed to find the closed form of the finite sum above.

$$\sum_{\substack{k=0\\k\neq m}}^{\infty} \frac{\begin{bmatrix}2k\\k\end{bmatrix}_{q^2}}{[k-m]_{q^2}} \cdot \frac{q^k}{(-q;q)_{2k}} = \frac{\begin{bmatrix}2m\\m\end{bmatrix}_{q^2} q^m \left(1-q^2\right)}{(-q;q)_{2m}} \sum_{k=2m+1}^{\infty} \frac{(-1)^{k-1} q^k}{1-q^k}.$$

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$$\sum_{k=0}^{m-1} \frac{\binom{2k}{k}}{4^k(2k-2m)} = ?$$

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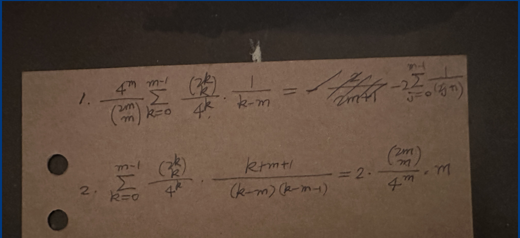

Karl's Idea

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Study a family of the finite sums: for $\nu \in \mathbb{N}$

$$S_\nu(m) := \sum_{k=0}^{m-1} \frac{\binom{2k}{k}}{4^k} \cdot \frac{(k+m+1)\cdots(k+m+\nu-1)}{(k-m)(k-m-1)\cdots(k-m-\nu+1)}.$$

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Proposition (S. Chern, K. Dilcher, and L.J.)

$$S_\nu(m) = \frac{\binom{2m}{m}}{4^m} \sum_{j=0}^{\nu-1} (-1)^{\nu-1-j} \binom{2m+\nu-1+j}{\nu-1} \binom{\nu-1}{j}$$
$$\left(\frac{(2m+1)(2m+3)\cdots(2m+2j-1)}{(2m+2)(2m+4)\cdots(2m+2j)} \sum_{s=0}^{m+j-1} \frac{2}{2s+1} \right.$$
$$\left. - \sum_{\ell=0}^{j-1} \frac{(2m+1)(2m+3)\cdots(2m+2\ell-1)}{(2m+2)(2m+4)\cdots(2m+2\ell)(\ell-j)} \right).$$

Examples

$$S_2(m) = \frac{\binom{2m}{m}}{4^m} \cdot 2m,$$

$$S_4(m) = \frac{\binom{2m}{m}}{4^m} \cdot \frac{2m}{3^2} \left(2m^2 + 3m + 4\right),$$

$$S_6(m) = \frac{\binom{2m}{m}}{4^m} \cdot \frac{2m}{2 \cdot 3^2 \cdot 5^2} \left(12m^4 + 60m^3 + 125m^2 + 125m + 128\right),$$

$$S_8(m) = \frac{\binom{2m}{m}}{4^m} \cdot \frac{2m}{2 \cdot 3^2 \cdot 5^2 \cdot 7^2} \left(40m^6 + 420m^5 + 1834m^4 + 4263m^3 \right.$$
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$$S_{2\ell+1}(m) = -\frac{\binom{2m+2\ell}{m+\ell}}{2^{2m+\ell}\binom{2\ell}{\ell}} \sum_{j=1}^{m} \binom{j+\ell-1}{\ell}^2 \left(\frac{1}{2j-1} + \frac{1}{2j+2\ell-1}\right).$$

q-analog

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Definition

$$S_{\nu,q}(m) := \sum_{k=0}^{m-1} \frac{\begin{bmatrix} 2k \\ k \end{bmatrix}_{q^2}}{(-q;q)_{2k}} \cdot \frac{q^k [k+m+1]_{q^2} \cdots [k+m+\nu-1]_{q^2}}{[k-m]_{q^2}[k-m-1]_{q^2} \cdots [k-m-\nu+1]_{q^2}}.$$

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Proposition (S. Chern, K. Dilcher, and L.J.)

For all $m \in \mathbb{N}$,

$$S_{1,q}(m) := \sum_{k=0}^{m-1} \frac{\begin{bmatrix}2k\\k\end{bmatrix}_{q^2}}{(-q;q)_{2k}} \frac{q^k}{[k-m]_{q^2}} = -\frac{\begin{bmatrix}2m\\m\end{bmatrix}_{q^2} q^m(1-q^2)}{(-q;q)_{2m}} \sum_{j=0}^{m-1} \frac{q^{2j+1}}{1-q^{2j+1}}.$$

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Remark

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Lemma (S. Chern, K. Dilcher, and L.J.)

For $\nu \geq 1$ and $k \neq m, m+1, \ldots, m+\nu-1$,

$$\sum_{j=0}^{\nu-1} \frac{(-1)^{\nu-1-j} q^{(\nu-1-j)(\nu-j)}}{1-q^{2k-2m-2j}} \begin{bmatrix} 2m+\nu-1+j \\ \nu-1 \end{bmatrix}_{q^2} \begin{bmatrix} \nu-1 \\ j \end{bmatrix}_{q^2} = \frac{(q^{2(k+m+1)}; q^2)_{\nu-1}}{(q^{2(k-m-\nu+1)}; q^2)_{\nu}}.$$

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Corollary (S. Chern, K. Dilcher, and L.J.)

$$\frac{[k+m+1]_{q^2} \cdots [k+m+\nu-1]_{q^2}}{[k-m]_{q^2}[k-m-1]_{q^2} \cdots [k-m-\nu+1]_{q^2}} = \sum_{j=0}^{\nu-1} \frac{(-1)^{\nu-1-j} q^{(\nu-1-j)(\nu-j)}}{[k-m-j]_{q^2}} \begin{bmatrix} 2m+\nu-1+j \\ \nu-1 \end{bmatrix}_{q^2} \begin{bmatrix} \nu-1 \\ j \end{bmatrix}_{q^2}.$$

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Theorem (S. Chern, K. Dilcher, and L.J.)

$$S_{\nu,q}(m) = -\frac{\begin{bmatrix} 2m \\ m \end{bmatrix}_{q^2} q^m}{(-q;q)_{2m}} \sum_{j=0}^{\nu-1} (-1)^{\nu-1-j} q^{(\nu-1-j)(\nu-j)} \begin{bmatrix} 2m+\nu-1+j \\ \nu-1 \end{bmatrix}_{q^2} \begin{bmatrix} \nu-1 \\ j \end{bmatrix}_{q^2} \times \left(\frac{(q^{2m+1}; q^2)_j \, q^j (1-q^2)}{(q^{2m+2}; q^2)_j} \sum_{s=0}^{m+j-1} \frac{q^{2s+1}}{1-q^{2s+1}} + \sum_{\ell=0}^{j-1} \frac{(q^{2m+1}; q^2)_\ell \, q^\ell}{(q^{2m+2}; q^2)_\ell \, [\ell-j]_{q^2}} \right).$$

Example: $n = 2$

$$S_{2,q}(m) = \sum_{k=0}^{m-1} \frac{\begin{bmatrix} 2k \\ k \end{bmatrix}_{q^2}}{[k-m]_{q^2}} \cdot \frac{q^k}{(-q;q)_{2k}} \cdot \frac{[k+m+1]_{q^2}}{[k-m-1]_{q^2}}$$

$$= \frac{\begin{bmatrix} 2m \\ m \end{bmatrix}_{q^2} q^{m+1}}{(-q;q)_{2m}} \left(\frac{q(1-q^{2m})(1+q^{2m+2})}{1-q^2} \right.$$

$$\left. -(1-q)(1-q^{2m+1}) \sum_{j=0}^{m-1} \frac{q^{2j+1}}{1-q^{2j+1}} \right).$$

"Squared"

Proposition (S. Chern, K. Dilcher, and L.J.)

$$\sum_{k=0}^{m-1} \frac{\binom{2k}{k}^2}{4^{2k}} \cdot \frac{1}{k+1} = 4m \cdot \frac{\binom{2m}{m}^2}{4^{2m}},$$

$$\sum_{k=0}^{m-1} \frac{\binom{2k}{k}^2}{4^{2k}} \cdot \frac{(2k+1)(4k+3)}{k+1} = 4m(2m+1) \cdot \frac{\binom{2m}{m}^2}{4^{2m}},$$

$$\sum_{k=0}^{m-1} \frac{\binom{2k}{k}^2}{4^{2k}} \cdot \frac{(2k+1)(2k+3)^2(4k+5)}{k+2}$$

$$= \frac{4m(2m+1)(2m+3)\left(2m^2+3m+4\right)}{3(m+1)} \cdot \frac{\binom{2m}{m}^2}{4^{2m}}.$$

with two q-analogues

$$\sum_{k=0}^{m-1} \frac{{\begin{bmatrix}2k\\k\end{bmatrix}}_{q^2}^2}{(-q;q)_{2k}^2} \frac{q^{2k}}{[2k+2]_q} = \frac{{\begin{bmatrix}2m\\m\end{bmatrix}}_{q^2}^2}{(-q;q)_{2m}^2}[2m]_q.$$

$$\sum_{k=0}^{m-1} \frac{{\begin{bmatrix}2k\\k\end{bmatrix}}_{q^2}^2}{(-q;q)_{2k}^2} \frac{q^{-k-1}[2k+1]_q[4k+3]_q}{[2k+2]_q} = \frac{{\begin{bmatrix}2m\\m\end{bmatrix}}_{q^2}^2}{(-q;q)_{2m}^2} q^{-m}[2m]_q[2m+1]_q.$$

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- There is a family of infinitely many "squared" identities.
- This is still open:

$$\sum_{k=0}^{n} \frac{\begin{bmatrix} 2k \\ k \end{bmatrix}_{q^2}}{\begin{bmatrix} \frac{2k+1+z}{2} \end{bmatrix}_{q^2}} \cdot \frac{q^k}{(-q;q)_{2k}} =?$$

The End

Thank you